Exercise 5.1 Question 1:

Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, at $x = -3$ and at $x = 5$.

Answer

The given function is $f(x) = 5x - 3$

At $x = 0$, $f(0) = 5 \times 0 - 3 = 3$

$\lim_{x \to 0} f(x) = \lim_{x \to 0}(5x - 3) = 5 \times 0 - 3 = -3$

$\therefore \lim_{x \to 0} f(x) = f(0)$

Therefore, f is continuous at x = 0

At $x = -3$, $f(-3) = 5 \times (-3) - 3 = -18$

$\lim_{x \to -3} f(x) = \lim_{x \to -3}(5x - 3) = 5 \times (-3) - 3 = -18$

$\therefore \lim_{x \to -3} f(x) = f(-3)$

Therefore, f is continuous at x = −3

At $x = 5$, $f(x) = f(5) = 5 \times 5 - 3 = 25 - 3 = 22$

$\lim_{x \to 5} f(x) = \lim_{x \to 5}(5x - 3) = 5 \times 5 - 3 = 22$

$\therefore \lim_{x \to 5} f(x) = f(5)$

Therefore, f is continuous at x = 5

Question 2:

Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$

Answer

Thus, f is continuous at x = 3

Question 3:

Examine the following functions for continuity.

(a) $f(x)=x-5$ (b) $f(x)=\frac{1}{x-5}, x \neq 5$

(c) $f(x)=\frac{x^2-25}{x+5}, x \neq -5$ (d) $f(x)=|x-5|$

Answer

(a) The given function is $f(x)=x-5$

It is evident that f is defined at every real number k and its value at k is k − 5.

It is also observed that, $\lim_{x\to k} f(x)=\lim_{x\to k}(x-5)=k-5=f(k)$

Hence, f is continuous at every real number and therefore, it is a continuous function.

(b) The given function is $f(x)=\frac{1}{x-5}, x \neq 5$

For any real number k ≠ 5, we obtain

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

(c) The given function is $f(x)=\frac{x^2-25}{x+5}, x \neq -5$

For any real number c ≠ −5, we obtain

$$\lim_{x\to c} f(x) = \lim_{x\to c} \frac{x^2-25}{x+5} = \lim_{x\to c} \frac{(x+5)(x-5)}{x+5} = \lim_{x\to c}(x-5) = (c-5)$$

$$\text{Also, } f(c) = \frac{(c+5)(c-5)}{c+5} = (c-5) \quad (\text{as } c \neq -5)$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

(d) The given function is $f(x) = |x-5| = \begin{cases} 5-x, \text{ if } x<5 \\ x-5, \text{ if } x \geq 5 \end{cases}$

This function f is defined at all points of the real line.

Let c be a point on a real line. Then, c < 5 or c = 5 or c > 5

Case I: c < 5

Then, f (c) = 5 − c

Therefore, f is continuous at all real numbers less than 5.

Case II : c = 5

Then, $f(c) = f(5) = (5-5) = 0$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(5-x) = 5-c$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at x = 5

Case III: c > 5

Therefore, f is continuous at all real numbers greater than 5.

Hence, f is continuous at every real number and therefore, it is a continuous function.

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(5) = 5$

Question 4:

$\therefore \lim_{x\to1^-} f(x) \neq \lim_{x\to1^+} f(x)$

Prove that the function $f(x) = x^n$ is continuous at x = n, where n is a positive integer.

Answer

The given function is f (x) = x^n

It is evident that f is defined at all positive integers, n, and its value at n is n^n.

Therefore, f is continuous at n, where n is a positive integer.

Question 5:

Is the function f defined by

continuous at x = 0? At x = 1? At x = 2?

Answer

The given function f is $f(x) = \begin{cases} x, & \text{if } x \le 1 \\ 5, & \text{if } x > 1 \end{cases}$

At x = 0,

It is evident that f is defined at 0 and its value at 0 is 0.

Therefore, f is continuous at x = 0

At x = 1,

f is defined at 1 and its value at 1 is 1.

The left hand limit of f at x = 1 is,

The right hand limit of f at x = 1 is,

Therefore, f is not continuous at x = 1

At x = 2,

f is defined at 2 and its value at 2 is 5.

Therefore, f is continuous at x = 2

Then, $\lim_{x\to 2} f(x) = \lim_{x\to 2}(5) = 5$

$\lim_{x\to 2} f(x) = f(2)$

Question 6:

Find all points of discontinuity of f, where f is defined by

Answer

It is evident that the given function f is defined at all the points of the real line.

Let c be a point on the real line. Then, three cases arise.

(i) c < 2

(ii) c > 2

(iii) c = 2

Case (i) c < 2

Therefore, f is continuous at all points x, such that x < 2

Case (ii) c > 2

$$\text{Then, } f(c) = 2c - 3$$

$$\lim_{x \to c} f(x) = \lim_{x \to c}(2x - 3) = 2c - 3$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that x > 2

Case (iii) c = 2

Then, the left hand limit of f at x = 2 is,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-}(2x + 3) = 2 \times 2 + 3 = 7$$

The right hand limit of f at x = 2 is,

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+}(2x - 3) = 2 \times 2 - 3 = 1$$

It is observed that the left and right hand limit of f at x = 2 do not coincide.

Therefore, f is not continuous at x = 2

Hence, x = 2 is the only point of discontinuity of f.

Question 7:

Find all points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} |x| + 3, \text{if } x \le -3 \\ -2x, \text{if } -3 < x < 3 \\ 6x + 2, \text{if } x \ge 3 \end{cases}$$

Answer

$$f(x) = \begin{cases} |x| + 3 = -x + 3, \ \text{if } x \le -3 \\ -2x, \ \text{if } -3 < x < 3 \\ 6x + 2, \ \text{if } x \ge 3 \end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

$$\text{If } c < -3, \text{ then } f(c) = -c + 3$$

$$\lim_{x \to c} f(x) = \lim_{x \to c}(-x + 3) = -c + 3$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that x < −3

Case II:

$$f(x)=\begin{cases}\frac{|x|}{x} & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$

$$\text{If } c=-3,\ \text{then } f(-3)=-(-3)+3=6$$

Therefore, f is continuous at x = −3

Case III:

$$\lim_{x\to -3^-} f(x)=\lim_{x\to -3^-}(-x+3)=-(-3)+3=6$$
$$\lim_{x\to -3^+} f(x)=\lim_{x\to -3^-}(-2x)=-2\times(-3)=6$$
$$\therefore \lim_{x\to -3} f(x)=f(-3)$$

Therefore, f is continuous in (−3, 3).

Case IV:

If c = 3, then the left hand limit of f at x = 3 is,

The right hand limit of f at x = 3 is,

It is observed that the left and right hand limit of f at x = 3 do not coincide.

Therefore, f is not continuous at x = 3

Case V:

Therefore, f is continuous at all points x, such that x > 3

Hence, x = 3 is the only point of discontinuity of f.

Question 8:

Find all points of discontinuity of f, where f is defined by

Answer

If $c > 0$, then $f(c) = 1$

$$\lim_{x \to c} f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

It is known that, $x < 0 \Rightarrow |x| = -x$ and $x > 0 \Rightarrow |x| = x$

Therefore, the given function can be rewritten as

$$f(x) = \begin{cases} \frac{|x|}{x} = \frac{-x}{x} = -1 \text{ if } x < 0 \\ 0, \text{ if } x = 0 \\ \frac{|x|}{x} = \frac{x}{x} = 1, \text{ if } x > 0 \end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

Therefore, f is continuous at all points x < 0

Case II:

If c = 0, then the left hand limit of f at x = 0 is,

The right hand limit of f at x = 0 is,

It is observed that the left and right hand limit of f at x = 0 do not coincide.

Therefore, f is not continuous at x = 0

Case III:

Therefore, f is continuous at all points x, such that x > 0

$$f(x)=\begin{cases} x+1, & \text{if } x \geq 1 \\ x^2+1, & \text{if } x<1 \end{cases}$$

Hence, x = 0 is the only point of discontinuity of f.

Question 9:

$$\therefore \lim f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2+1, & \text{if } x<1 \end{cases}$$

Find all points of discontinuity of f, where f is defined by

Answer

$$f(x)=\begin{cases} \dfrac{x}{|x|}, & \text{if } x<0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

It is known that, $x<0 \Rightarrow |x|=-x$

Therefore, the given function can be rewritten as

Let c be any real number. Then, $\lim_{x\to c} f(x) = \lim_{x\to c}(-1) = -1$

Also, $f(c) = -1 = \lim_{x\to c} f(x)$

Therefore, the given function is a continuous function.

Hence, the given function has no point of discontinuity.

Question 10:

Find all points of discontinuity of f, where f is defined by

Answer

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

Therefore, f is continuous at all points x, such that x < 1

Case II:

The left hand limit of f at x = 1 is,

The right hand limit of f at x = 1 is,

Therefore, f is continuous at x = 1

Case III:

Therefore, f is continuous at all points x, such that x > 1

Hence, the given function f has no point of discontinuity.

Question 11:

Find all points of discontinuity of f, where f is defined by

$$f(x)=\begin{cases} x^{10}-1, & \text{if } x\le 1 \\ x^2, & \text{if } x>1 \end{cases}$$

$$\begin{cases} x^3-3, & \text{if } x\le 2 \\ x^2+1, & \text{if } x>2 \end{cases}$$

Answer

$$f(x)=\begin{cases} x^{10}-1, & \text{if } x\le 1 \\ x^2, & \text{if } x>1 \end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c<1$, then $f(c)=c^{10}-1$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}\left(x^{10}-1\right)=c^{10}-1$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that x < 2

Case II:

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\left(x^{10}-1\right)=1^{10}-1=1-1=0$$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\left(x^2\right)=1^2=1$$

Therefore, f is continuous at x = 2

Case III:

If $c>1$, then $f(c)=c^2$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}\left(x^2\right)=c^2$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that x > 2

Thus, the given function f is continuous at every point on the real line.

Hence, f has no point of discontinuity.

Question 12:

Find all points of discontinuity of f, where f is defined by

Answer $f(x)=\begin{cases} x+5, \text{ if } x\le 1 \\ x-5, \text{ if } x>1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

Therefore, f is continuous at all points x, such that x < 1

Case II:

If c = 1, then the left hand limit of f at x = 1 is,

The right hand limit of f at x = 1 is,

It is observed that the left and right hand limit of f at x = 1 do not coincide.

Therefore, f is not continuous at x = 1

Case III:

Therefore, f is continuous at all points x, such that x > 1

Thus, from the above observation, it can be concluded that x = 1 is the only point of discontinuity of f.

Question 13:

Is the function defined by

a continuous function?

Answer

The given function is $f(x)=\begin{cases} x+5, & \text{if } x\le 1 \\ x-5, & \text{if } x>1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

$\text{If } c<1, \text{ then } f(c)=c+5 \text{ and } \lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(x+5)=c+5$

Therefore, f is continuous at all points x, such that x < 1

Case II:

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

The left hand limit of f at x = 1 is,

$\text{If } c=1, \text{ then } f(1)=1+5=6$

The right hand limit of f at x = 1 is,

$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}(x+5)=1+5=6$

It is observed that the left and right hand limit of f at x = 1 do not coincide.

Therefore, f is not continuous at x = 1

Case III:

Therefore, f is continuous at all points x, such that x > 1

Thus, from the above observation, it can be concluded that x = 1 is the only point of discontinuity of f.

Question 14:

Discuss the continuity of the function f, where f is defined by

$$f(x)=\begin{cases}3, & \text{if } 0\le x\le 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\le x\le 10\end{cases}$$

Answer

The given function is $f(x)=\begin{cases}3, & \text{if } 0\le x\le 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\le x\le 10\end{cases}$

The given function is defined at all points of the interval [0, 10].

Let c be a point in the interval [0, 10].

Case I:

$$\text{If } 0\le c<1, \text{ then } f(c)=3 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(3)=3$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous in the interval [0, 1).

Case II:

$$\text{If } c=1, \text{ then } f(3)=3$$

The left hand limit of f at x = 1 is,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(3)=3$$

The right hand limit of f at x = 1 is,

It is observed that the left and right hand limits of f at x = 1 do not coincide.

Therefore, f is not continuous at x = 1

Case III:

Therefore, f is continuous at all points of the interval (1, 3).

Case IV:

The left hand limit of f at x = 3 is,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (4) = 4$$

The right hand limit of f at x = 3 is,

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (5) = 5$$

It is observed that the left and right hand limits of f at x = 3 do not coincide.

Therefore, f is not continuous at x = 3

Case V:

$$\lim_{x \to c} f(x) = f(c)$$

$$\text{If } 3 < c \le 10, \text{ then } f(c) = 5 \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c} (5) = 5$$

Therefore, f is continuous at all points of the interval (3, 10].

Hence, f is not continuous at x = 1 and x = 3

Question 15:

Discuss the continuity of the function f, where f is defined by

Answer

The given function is $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$

The given function is defined at all points of the real line.

Let c be a point on the real line.

Case I:

Therefore, f is continuous at all points x, such that x < 0

Case II:

$$\text{If } c = 0, \text{ then } f(c) = f(0) = 0$$

The left hand limit of f at x = 0 is,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x) = 2 \times 0 = 0$$

The right hand limit of f at x = 0 is,

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (0) = 0$$

$$\therefore \lim_{x \to 0} f(x) = f(0)$$

Therefore, f is continuous at x = 0

Case III:

$$\text{If } 0 < c < 1, \text{ then } f(x) = 0 \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c} (0) = 0$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points of the interval (0, 1).

Case IV:

The left hand limit of f at x = 1 is,

The right hand limit of f at x = 1 is,

It is observed that the left and right hand limits of f at x = 1 do not coincide.

Therefore, f is not continuous at x = 1

Case V:

Therefore, f is continuous at all points x, such that x > 1

Hence, f is not continuous only at x = 1

Question 16:

Discuss the continuity of the function f, where f is defined by

$$f(x)=\begin{cases}-2, & \text{if } x\le -1\\ 2x, & \text{if } -1<x\le 1\\ 2, & \text{if } x>1\end{cases}$$

Answer

$$f(x)=\begin{cases}-2, & \text{if } x\le -1\\ 2x, & \text{if } -1<x\le 1\\ 2, & \text{if } x>1\end{cases}$$

The given function is defined at all points of the real line.

Let c be a point on the real line.

Case I:

If $c<-1$, then $f(c)=-2$ and $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(-2)=-2$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that x < −1

Case II:

If $c=-1$, then $f(c)=f(-1)=-2$

The left hand limit of f at x = −1 is,

$\lim\limits_{x\to -1^-} f(x)=\lim\limits_{x\to -1^-}(-2)=-2$

The right hand limit of f at x = −1 is,

Therefore, f is continuous at x = −1

Case III:

Therefore, f is continuous at all points of the interval (−1, 1).

Case IV:

The left hand limit of f at x = 1 is,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) \quad ...(1)$$

Also,

$$\lim_{x\to 3} f(x) = \lim_{x\to 3}(ax+1) = 3a+1$$

The right hand limit of f at x = 1 is,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(2x) = 2\times 1 = 2 \; 3$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2 = 2$$

Therefore, f is continuous at x = 2

Case V: $\therefore \lim_{x\to 1} f(x) = f(c)$

If $c > 1$, then $f(c) = 2$ and $\lim_{x\to c} f(x) = \lim_{x\to c}(2) = 2$

$\lim_{x\to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that x > 1

Thus, from the above observations, it can be concluded that f is continuous at all points of the real line.

Question 17:

Find the relationship between a and b so that the function f defined by

is continuous at x = 3.

Answer

If f is continuous at x = 3, then

Therefore, from (1), we obtain

$$3a+1=3b+3=3a+1$$
$$\Rightarrow 3a+1=3b+3$$
$$\Rightarrow 3a=3b+2$$
$$\Rightarrow a=b+\frac{2}{3}$$

Therefore, the required relationship is given by, $a=b+\frac{2}{3}$

Question 18:

For what value of λ is the function defined by

continuous at x = 0? What about continuity at x = 1?

Answer

The given function is $f(x)=\begin{cases}\lambda\left(x^2-2x\right), & \text{if } x\le 0\\ 4x+1, & \text{if } x>0\end{cases}$

If f is continuous at x = 0, then

Therefore, there is no value of λ for which f is continuous at x = 0

At x = 1,

f (1) = 4x + 1 = 4 × 1 + 1 = 5

Therefore, for any values of λ, f is continuous at x = 1

$\lim(4x+1)=4\times1+1=5$

$\therefore \lim_{x\to1} f(x)=f(1)$

Question 19:

Show that the function defined by $g(x)=x-[x]$ is discontinuous at all integral point.

Here $[x]$ denotes the greatest integer less than or equal to x.

Answer

The given function is $g(x)=x-[x]$

It is evident that g is defined at all integral points.

Let n be an integer.

Then,

The left hand limit of f at x = n is,

The right hand limit of f at x = n is,

It is observed that the left and right hand limits of f at x = n do not coincide.

Therefore, f is not continuous at x = n

Hence, g is discontinuous at all integral points.

Question 20:

Is the function defined by $f(x)=x^2-\sin x+5$ continuous at x = p?

Answer

The given function is $f(x)=x^2-\sin x+5$

It is evident that f is defined at x = p

$\text{At } x=\pi,\ f(x)=f(\pi)=\pi^2-\sin\pi+5=\pi^2-0+5=\pi^2+5$

Consider $\lim\limits_{x\to\pi} f(x)=\lim\limits_{x\to\pi}\left(x^2-\sin x+5\right)$

Put $x=\pi+h$

If $x\to\pi$, *then it is evident that* $h\to 0$

$$\begin{aligned}
\therefore \lim_{x\to\pi} f(x) &= \lim_{x\to\pi}\left(x^2-\sin x+5\right)\\
&= \lim_{h\to 0}\left[(\pi+h)^2-\sin(\pi+h)+5\right]\\
&= \lim_{h\to 0}(\pi+h)^2-\lim_{h\to 0}\sin(\pi+h)+\lim_{h\to 0}5\\
&= (\pi+0)^2-\lim_{h\to 0}\left[\sin\pi\cos h+\cos\pi\sin h\right]+5\\
&= \pi^2-\lim_{h\to 0}\sin\pi\cos h-\lim_{h\to 0}\cos\pi\sin h+5\\
&= \pi^2-\sin\pi\cos 0-\cos\pi\sin 0+5\\
&= \pi^2-0\times 1-(-1)\times 0+5\\
&= \pi^2+5
\end{aligned}$$

$\therefore \lim\limits_{x\to\pi} f(x)=f(\pi)$

Therefore, the given function f is continuous at x = π

Question 21:

Discuss the continuity of the following functions.

(a) f (x) = sin x + cos x

(b) f (x) = sin x − cos x

(c) f (x) = sin x × cos x

Answer

It is known that if g and h are two continuous functions, then

$g+h,\ g-h,$ and $g.h$ are also continuous.

It has to proved first that g (x) = sin x and h (x) = cos x are continuous functions.

Let g (x) = sin x

It is evident that g (x) = sin x is defined for every real number.

Let c be a real number. Put x = c + h

If x → c, then h → 0

Therefore, g is a continuous function.

Let h (x) = cos x

It is evident that h (x) = cos x is defined for every real number.

Let c be a real number. Put x = c + h

If x → c, then h → 0

h (c) = cos c

$$g(c)=\sin c$$

$$\begin{aligned}\lim_{x\to c} g(x) &= \lim_{x\to c}\sin x\\ &= \lim_{h\to 0}\sin(c+h)\\ &= \lim_{h\to 0}\left[\sin c\cos h+\cos c\sin h\right]\\ &= \lim_{h\to 0}(\sin c\cos h)+\lim_{h\to 0}(\cos c\sin h)\\ &= \sin c\cos 0+\cos c\sin 0\\ &= \sin c+0\\ &= \sin c\end{aligned}$$

$$\therefore\ \lim_{x\to c} g(x)=g(c)$$

Therefore, h is a continuous function.

Therefore, it can be concluded that

(a) f (x) = g (x) + h (x) = sin x + cos x is a continuous function

(b) f (x) = g (x) − h (x) = sin x − cos x is a continuous function

(c) f (x) = g (x) × h (x) = sin x × cos x is a continuous function

Question 22:

Discuss the continuity of the cosine, cosecant, secant and cotangent functions,

Answer

It is known that if g and h are two continuous functions, then

$$(i)\ \frac{h(x)}{g(x)},\ g(x)\neq 0 \text{ is continuous}$$

$$(ii)\ \frac{1}{g(x)},\ g(x)\neq 0 \text{ is continuous}$$

$$(iii)\ \frac{1}{h(x)},\ h(x)\neq 0 \text{ is continuous}$$

It has to be proved first that g (x) = sin x and h (x) = cos x are continuous functions.

Let g (x) = sin x

It is evident that g (x) = sin x is defined for every real number.

Let c be a real number. Put x = c + h

If x $\rightarrow$ c, then h $\rightarrow$ 0

Therefore, g is a continuous function.

Let h (x) = cos x

It is evident that h (x) = cos x is defined for every real number.

Let c be a real number. Put x = c + h

If x $\rightarrow$ c, then h $\rightarrow$ 0

h (c) = cos c

$$\lim_{x \to c} h(x) = \lim_{x \to c} \cos x$$
$$= \lim_{h \to 0} \cos(c+h)$$
$$= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h]$$
$$= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h$$
$$= \cos c \cos 0 - \sin c \sin 0$$
$$= \cos c \times 1 - \sin c \times 0$$
$$= \cos c$$
$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, h (x) = cos x is continuous function.

It can be concluded that,

$$\operatorname{cosec} x = \frac{1}{\sin x},\ \sin x \neq 0 \text{ is continuous}$$
$$\Rightarrow \operatorname{cosec} x,\ x \neq n\pi\ (n \in Z) \text{ is continuous}$$

Therefore, cosecant is continuous except at x = np, n Î Z

$$\sec x = \frac{1}{\cos x},\ \cos x \neq 0 \text{ is continuous}$$
$$\Rightarrow \sec x,\ x \neq (2n+1)\frac{\pi}{2}\ (n \in \mathbf{Z}) \text{ is continuous}$$

Therefore, secant is continuous except at $x = (2n+1)\frac{\pi}{2}\ (n \in \mathbf{Z})$

Therefore, cotangent is continuous except at x = np, n Î Z

Question 23:

Find the points of discontinuity of f, where

Answer

$$f(x)=\begin{cases}\dfrac{\sin x}{x}, \text{ if } x<0\\ x+1, \text{ if } x\geq 0\end{cases}$$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

$$\text{If } c<0, \text{ then } f(c)=\frac{\sin c}{c} \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}\left(\frac{\sin x}{x}\right)=\frac{\sin c}{c}$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that x < 0

Case II:

Therefore, f is continuous at all points x, such that x > 0

Case III:

The left hand limit of f at x = 0 is,

The right hand limit of f at x = 0 is,

Therefore, f is continuous at x = 0

From the above observations, it can be concluded that f is continuous at all points of the real line.

Thus, f has no point of discontinuity.

Question 24:

Determine if f defined by

$$f(x)=\begin{cases}x^2\sin\frac{1}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$

is a continuous function?

Answer

$$f(x)=\begin{cases}x^2\sin\frac{1}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

Therefore, f is continuous at all points x ≠ 0

Case II:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right)$$

It is known that, $-1 \le \sin\frac{1}{x} \le 1,\ x \ne 0$

$$\Rightarrow -x^2 \le \sin\frac{1}{x} \le x^2$$

$$\Rightarrow \lim_{x \to 0}\left(-x^2\right) \le \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) \le \lim_{x \to 0} x^2$$

$$\Rightarrow 0 \le \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) \le 0$$

$$\Rightarrow \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$$

$$\therefore \lim_{x \to 0^-} f(x) = 0$$

Similarly, $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$

$$\therefore \lim_{x \to 0^-} f(x) = f(0) = \lim_{x \to 0^+} f(x)$$

Therefore, f is continuous at x = 0

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Question 25:

Examine the continuity of f, where f is defined by

Answer

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

Therefore, f is continuous at all points x, such that x ≠ 0

Case II:

$$\text{If } c \neq 0, \text{ then } f(c) = \sin c - \cos c$$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (\sin x - \cos x) = \sin c - \cos c$$

$$\therefore \lim f(x) = f(c)$$

$$\text{If } c = 0, \text{ then } f(0) = -1$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} (\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} (\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$$

$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

Therefore, f is continuous at x = 0

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Question 26:

Find the values of k so that the function f is continuous at the indicated point.

Answer

The given function f is continuous at $x = \frac{\pi}{2}$, if f is defined at $x = \frac{\pi}{2}$ and if the value of the f at $x = \frac{\pi}{2}$ equals the limit of f at $x = \frac{\pi}{2}$.

It is evident that f is defined at $x=\frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right)=3$

$$\lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}$$

Put $x=\frac{\pi}{2}+h$

Then, $x\to\frac{\pi}{2}\Rightarrow h\to 0$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}=\lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}$$

$$=k\lim_{h\to 0}\frac{-\sin h}{-2h}=\frac{k}{2}\lim_{h\to 0}\frac{\sin h}{h}=\frac{k}{2}.1=\frac{k}{2}$$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x)=f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{k}{2}=3$$

$$\Rightarrow k=6$$

Therefore, the required value of k is 6.

Question 27:

Find the values of k so that the function f is continuous at the indicated point.

Answer

The given function is $f(x)=\begin{cases}kx^2, & \text{if } x\le 2\\ 3, & \text{if } x>2\end{cases}$

The given function f is continuous at x = 2, if f is defined at x = 2 and if the value of f at x = 2 equals the limit of f at x = 2

It is evident that f is defined at x = 2 and $f(2)=k(2)^2=4k$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{x\to 2^-}\left(kx^2\right) = \lim_{x\to 2^+}(3) = 4k$$

$$\Rightarrow k \times 2^2 = 3 = 4k$$

$$\Rightarrow 4k = 3 = 4k$$

$$\Rightarrow 4k = 3$$

$$\Rightarrow k = \frac{3}{4}$$

Therefore, the required value of k is $\frac{3}{4}$.

Question 28:

Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} kx+1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases} \qquad \text{at } x = \pi$$

Answer

The given function is $f(x) = \begin{cases} kx+1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$

The given function f is continuous at x = p, if f is defined at x = p and if the value of f at x = p equals the limit of f at x = p

It is evident that f is defined at x = p and $f(\pi) = k\pi + 1$

Therefore, the required value of k is $-\frac{2}{\pi}$.

Question 29:

Find the values of k so that the function f is continuous at the indicated point.

$$f(x)=\begin{cases} kx+1, & \text{if } x\le 5 \\ 3x-5, & \text{if } x>5 \end{cases} \qquad \text{at } x=5$$

Answer

$$f(x)=\begin{cases} kx+1, & \text{if } x\le 5 \\ 3x-5, & \text{if } x>5 \end{cases}$$

The given function f is continuous at x = 5, if f is defined at x = 5 and if the value of f at x = 5 equals the limit of f at x = 5

It is evident that f is defined at x = 5 and $f(5)=kx+1=5k+1$

$$\lim_{x\to 5^-} f(x)=\lim_{x\to 5^+} f(x)=f(5)$$

$$\Rightarrow \lim_{x\to 5^-}(kx+1)=\lim_{x\to 5^+}(3x-5)=5k+1$$

$$\Rightarrow 5k+1=15-5=5k+1$$

$$\Rightarrow 5k+1=10$$

$$\Rightarrow 5k=9$$

$$\Rightarrow k=\frac{9}{5}$$

Therefore, the required value of k is $\frac{9}{5}$.

Question 30:

Find the values of a and b such that the function defined by

is a continuous function.

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Answer

$$f(x)=\begin{cases}5, & \text{if } x\le 2\\ ax+b, & \text{if } 2<x<10\\ 21, & \text{if } x\ge 10\end{cases}$$

It is evident that the given function f is defined at all points of the real line.

If f is a continuous function, then f is continuous at all real numbers.

In particular, f is continuous at x = 2 and x = 10

Since f is continuous at x = 2, we obtain

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$$
$$\Rightarrow \lim_{x\to 2^-}(5)=\lim_{x\to 2^+}(ax+b)=5$$
$$\Rightarrow 5=2a+b=5$$
$$\Rightarrow 2a+b=5 \qquad \ldots(1)$$

Since f is continuous at x = 10, we obtain

$$\lim_{x\to 10^-} f(x)=\lim_{x\to 10^+} f(x)=f(10)$$
$$\Rightarrow \lim_{x\to 10^-}(ax+b)=\lim_{x\to 10^+}(21)=21$$
$$\Rightarrow 10a+b=21=21$$
$$\Rightarrow 10a+b=21 \qquad \ldots(2)$$

On subtracting equation (1) from equation (2), we obtain

8a = 16

⇒ a = 2

By putting a = 2 in equation (1), we obtain

2 × 2 + b = 5

⇒ 4 + b = 5

$\Rightarrow b = 1$

Therefore, the values of a and b for which f is a continuous function are 2 and 1 respectively.

Question 31:

Show that the function defined by f (x) = cos (x^2) is a continuous function.

Answer

The given function is f (x) = cos (x^2)

This function f is defined for every real number and f can be written as the composition of two functions as,

f = g o h, where g (x) = cos x and h (x) = x^2

It has to be first proved that g (x) = cos x and h (x) = x^2 are continuous functions.

It is evident that g is defined for every real number.

Let c be a real number.

Then, g (c) = cos c

Therefore, g (x) = cos x is continuous function.

h (x) = x^2

Clearly, h is defined for every real number.

Let k be a real number, then h (k) = k^2

$$\lim_{x\to k} h(x) = \lim_{x\to k} x^2 = k^2$$
$$\therefore \lim_{x\to k} h(x) = h(k)$$

Therefore, h is a continuous function.

It is known that for real valued functions g and h,such that (g o h) is defined at c, if g is continuous at c and if f is continuous at g (c), then (f o g) is continuous at c.

Therefore, $f(x) = (goh)(x) = \cos(x^2)$ is a continuous function.

Question 32:

Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

Answer

The given function is $f(x) = |\cos x|$

This function f is defined for every real number and f can be written as the composition of two functions as,

f = g o h, where $g(x) = |x|$ and $h(x) = \cos x$

It has to be first proved that $g(x) = |x|$ and $h(x) = \cos x$ are continuous functions.

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I:

If $c > 0$, then $g(c) = c$ and $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} x = c$

$\therefore \lim\limits_{x \to c} g(x) = g(c)$

If $c = 0$, then $g(c) = g(0) = 0$

$\lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^-} (-x) = 0$

$\lim\limits_{x \to 0^+} g(x) = \lim\limits_{x \to 0^+} (x) = 0$

$\therefore \lim\limits_{x \to 0} g(x) = \lim\limits_{x \to 0^+} (x) = g(0)$

Case II:

Therefore, g is continuous at all points x, such that x > 0

Case III:

Therefore, g is continuous at x = 0

From the above three observations, it can be concluded that g is continuous at all points.

h (x) = cos x

It is evident that h (x) = cos x is defined for every real number.

Let c be a real number. Put x = c + h

If x → c, then h → 0

h (c) = cos c

Therefore, h (x) = cos x is a continuous function.

It is known that for real valued functions g and h,such that (g o h) is defined at c, if g is continuous at c and if f is continuous at g (c), then (f o g) is continuous at c.

Therefore, $f(x)=(goh)(x)=g(h(x))=g(\cos x)=|\cos x|$ is a continuous function.

Question 33:

Examine that $\sin|x|$ is a continuous function.

Answer

Let $f(x)=\sin|x|$

This function f is defined for every real number and f can be written as the composition of two functions as,

f = g o h, where $g(x)=|x|$ and $h(x)=\sin x$

$\left[\because (goh)(x)=g(h(x))=g(\sin x)=|\sin x|=f(x)\right]$

It has to be proved first that $g(x)=|x|$ and $h(x)=\sin x$ are continuous functions.

$g(x)=|x|$ can be written as

$$g(x)=\begin{cases}-x, & \text{if } x<0\\ x, & \text{if } x\geq 0\end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I:

Therefore, g is continuous at all points x, such that x < 0

Case II:

Therefore, g is continuous at all points x, such that x > 0

Case III:

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} (-x) = 0$$
$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} (x) = 0$$
$$\therefore \lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} (x) = g(0)$$

Therefore, g is continuous at x = 0

From the above three observations, it can be concluded that g is continuous at all points.

h (x) = sin x

It is evident that h (x) = sin x is defined for every real number.

Let c be a real number. Put x = c + k

If x → c, then k → 0

h (c) = sin c

$$h(c) = \sin c$$
$$\begin{aligned}\lim_{x\to c} h(x) &= \lim_{x\to c} \sin x \\ &= \lim_{k\to 0} \sin(c+k) \\ &= \lim_{k\to 0} [\sin c \cos k + \cos c \sin k] \\ &= \lim_{k\to 0} (\sin c \cos k) + \lim_{h\to 0} (\cos c \sin k) \\ &= \sin c \cos 0 + \cos c \sin 0 \\ &= \sin c + 0 \\ &= \sin c\end{aligned}$$
$$\therefore \lim_{x\to c} h(x) = g(c)$$

Therefore, h is a continuous function.

It is known that for real valued functions g and h,such that (g o h) is defined at c, if g is continuous at c and if f is continuous at g (c), then (f o g) is continuous at c.

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\sin x) = |\sin x|$ is a continuous function.

Question 34:

Find all the points of discontinuity of f defined by $f(x) = |x| - |x+1|$.

Answer

The given function is $f(x) = |x| - |x+1|$

The two functions, g and h, are defined as

$$g(x)=|x| \text{ and } h(x)=|x+1|$$

Then, f = g − h

The continuity of g and h is examined first.

$g(x)=|x|$ can be written as

$$g(x)=\begin{cases}-x, & \text{if } x<0\\ x, & \text{if } x\ge 0\end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I:

If $c<0$, then $g(c)=-c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c}(-x)=-c$

$$\therefore \lim_{x\to c} g(x)=g(c)$$

Therefore, g is continuous at all points x, such that x < 0

Case II:

If $c>0$, then $g(c)=c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c} x=c$

$$\therefore \lim_{x\to c} g(x)=g(c)$$

Therefore, g is continuous at all points x, such that x > 0

Case III:

Therefore, g is continuous at x = 0

From the above three observations, it can be concluded that g is continuous at all points.

Clearly, h is defined for every real number.

Let c be a real number.

Case I:

$$\text{If } c<-1, \text{ then } h(c)=-(c+1) \text{ and } \lim_{x\to c} h(x)=\lim_{x\to c}\left[-(x+1)\right]=-(c+1)$$

$$\therefore \lim_{x\to c} h(x)=h(c)$$

Therefore, h is continuous at all points x, such that x < −1

Case II:

$$\text{If } c>-1, \text{ then } h(c)=c+1 \text{ and } \lim_{x\to c} h(x)=\lim_{x\to c}(x+1)=c+1$$

$$\therefore \lim_{x\to c} h(x)=h(c)$$

Therefore, h is continuous at all points x, such that x > −1

Case III:

$$\text{If } c=-1, \text{ then } h(c)=h(-1)=-1+1=0$$

$$\lim_{x\to -1^-} h(x)=\lim_{x\to -1^-}\left[-(x+1)\right]=-(-1+1)=0$$

$$\lim_{x\to -1^+} h(x)=\lim_{x\to -1^+}(x+1)=(-1+1)=0$$

$$\therefore \lim_{x\to -1^-} h(x)=\lim_{h\to -1^+} h(x)=h(-1)$$

Therefore, h is continuous at x = −1

From the above three observations, it can be concluded that h is continuous at all points of the real line.

g and h are continuous functions. Therefore, f = g − h is also a continuous function.

Therefore, f has no point of discontinuity.

Exercise 5.2

Question 1:

Differentiate the functions with respect to x.

Answer

$\sin\left(x^2+5\right)$

Let $f(x)=\sin\left(x^2+5\right)$, $u(x)=x^2+5$, and $v(t)=\sin t$

Then, $(vou)(x)=v(u(x))=v\left(x^2+5\right)=\tan\left(x^2+5\right)=f(x)$

Thus, f is a composite of two functions.

Put $t=u(x)=x^2+5$

Then, we obtain

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos\left(x^2+5\right)$$

$$\frac{dt}{dx}=\frac{d}{dx}\left(x^2+5\right)=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(5)=2x+0=2x$$

Therefore, by chain rule, $\frac{df}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}=\cos\left(x^2+5\right)\times 2x=2x\cos\left(x^2+5\right)$

Alternate method

$$\begin{aligned}\frac{d}{dx}\left[\sin\left(x^2+5\right)\right]&=\cos\left(x^2+5\right)\cdot\frac{d}{dx}\left(x^2+5\right)\\&=\cos\left(x^2+5\right)\cdot\left[\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(5)\right]\\&=\cos\left(x^2+5\right)\cdot[2x+0]\\&=2x\cos\left(x^2+5\right)\end{aligned}$$

Question 2:

Differentiate the functions with respect to x.

Answer

Thus, f is a composite function of two functions.

Put t = u (x) = sin x

$$\frac{d}{dx}\left[\sin(ax+b)\right]=\cos(ax+b).\frac{d}{dx}(ax+b)$$

s t

$$=\cos(ax+b)\cdot\left[\frac{d}{dx}(ax)+\frac{d}{dx}(b)\right])=f(x)$$

$$=\cos(ax+b).(a+0)$$

By chain rule, $\frac{df}{dx}=\frac{dv}{dt}\cdot\frac{dt}{dx}=-\sin(\sin x)\cdot\cos x=-\cos x\sin(\sin x)$

Alternate method

$$\therefore\frac{dv}{dt}=\frac{d}{dt}[\cos t]=-\sin t=-\sin(\sin x)$$

$$\frac{dt}{dx}=\frac{d}{dx}(\sin x)=\cos x$$

Question 3:

Differentiate the functions with respect to x.

Answer

Thus, f is a composite function of two functions, u and v.

Put t = u (x) = ax + b

Hence, by chain rule, we obtain

Alternate method

Question 4:

Differentiate the functions with respect to x.

Answer

Thus, f is a composite function of three functions, u, v, and w.

Hence, by chain rule, we obtain

$$\frac{dt}{dx}=\frac{dw}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{dx}$$
$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\times\sec^2\sqrt{x}\times\frac{1}{2\sqrt{x}}$$
$$=\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)$$
$$=\frac{\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method

$$\frac{d}{dx}\left[\sec\left(\tan\sqrt{x}\right)\right]=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right)\cdot\frac{d}{dx}\left(\tan\sqrt{x}\right)$$
$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\cdot\sec^2\left(\sqrt{x}\right)\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$
$$=\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\cdot\sec^2\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}$$
$$=\frac{\sec\left(\tan\sqrt{x}\right)\cdot\tan\left(\tan\sqrt{x}\right)\sec^2\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Question 5:

Differentiate the functions with respect to x.

Answer

The given function is $f(x)=\frac{\sin(ax+b)}{\cos(cx+d)}=\frac{g(x)}{h(x)}$, where g (x) = sin (ax + b) and h (x) = cos (cx + d)

∴ g is a composite function of two functions, u and v.

$$\text{Put } t = u(x) = ax + b$$

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax+b)$$

$$\frac{dt}{dx} = \frac{d}{dx}(ax+b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a$$

Therefore, by chain rule, we obtain

$$g' = \frac{dg}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(ax+b) \cdot a = a\cos(ax+b)$$

$$\text{Consider } h(x) = \cos(cx+d)$$

$$\text{Let } p(x) = cx + d,\ q(y) = \cos y$$

$$\text{Then, } (qop)(x) = q(p(x)) = q(cx+d) = \cos(cx+d) = h(x)$$

∴h is a composite function of two functions, p and q.

Put y = p (x) = cx + d

Therefore, by chain rule, we obtain

$$\therefore f' = \frac{a\cos(ax+b)\cdot\cos(cx+d) - \sin(ax+b)\{-c\sin(cx+d)\}}{[\cos(cx+d)]^2}$$

$$= \frac{a\cos(ax+b)}{\cos(cx+d)} + c\sin(ax+b)\cdot\frac{\sin(cx+d)}{\cos(cx+d)} \times \frac{1}{\cos(cx+d)}$$

$$= a\cos(ax+b)\sec(cx+d) + c\sin(ax+b)\tan(cx+d)\sec(cx+d)$$

Question 6:

Differentiate the functions with respect to x.

Answer

$\cos x^3.\sin^2(x^5)$

The given function is $\cos x^3.\sin^2(x^5)$

$$\frac{d}{dx}\left[\cos x^3\cdot\sin^2(x^5)\right] = \sin^2(x^5)\times\frac{d}{dx}(\cos x^3) + \cos x^3\times\frac{d}{dx}\left[\sin^2(x^5)\right]$$

$$= \sin^2(x^5)\times(-\sin x^3)\times\frac{d}{dx}(x^3) + \cos x^3\times 2\sin(x^5)\cdot\frac{d}{dx}\left[\sin x^5\right]$$

$$= -\sin x^3\sin^2(x^5)\times 3x^2 + 2\sin x^5\cos x^3\cdot\cos x^5\times\frac{d}{dx}(x^5)$$

$$= -3x^2\sin x^3\cdot\sin^2(x^5) + 2\sin x^5\cos x^5\cos x^3\cdot\times 5x^4$$

$$= 10x^4\sin x^5\cos x^5\cos x^3 - 3x^2\sin x^3\sin^2(x^5)$$

Question 7:

Differentiate the functions with respect to x.

Answer

$$\text{Then, } \frac{dt}{dx} = \frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x}}$$

$$\text{And, } \frac{dv}{dt} = \frac{d}{dt}(\cos t) = -\sin t$$

$$= -\sin\left(\sqrt{x}\right)$$

$$= -\sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}} \times \frac{1}{\sin^2\left(x^2\right)} \times (2x)$$

$$= \frac{-2x}{\sqrt{\cos x^2}\sqrt{\sin x^2}\,\sin x^2}$$

$$= \frac{-2\sqrt{2}\,x}{\sqrt{2\sin x^2\cos x^2}\,\sin x^2}$$

$$= \frac{-2\sqrt{2}\,x}{\sin x^2\sqrt{\sin 2x^2}}$$

Question 28

Differentiate the functions with respect to x.

Answer

Clearly, f is a composite function of two functions, u and v, such that

By using chain rule, we obtain

Alternate method

Question 9:

Prove that the function f given by

$f(x)=|x-1|,\ x\in\mathbf{R}$ is notdifferentiable at x = 1.

Answer

The given function is $f(x)=|x-1|,\ x\in\mathbf{R}$

It is known that a function f is differentiable at a point x = c in its domain if both

$\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h}$ and $\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at x = 1,

consider the left hand limit of f at x = 1

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{|1+h-1|-|1-1|}{h}$$

$$= \lim_{h\to 0^-} \frac{|h|-0}{h} = \lim_{h\to 0^-} \frac{-h}{h} \qquad (h<0 \Rightarrow |h| = -h)$$

$$= -1$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{|1+h-1|-|1-1|}{h}$$

$$= \lim_{h\to 0^+} \frac{|h|-0}{h} = \lim_{h\to 0^+} \frac{h}{h} \qquad (h>0 \Rightarrow |h| = h)$$

$$= 1$$

Since the left and right hand limits of f at x = 1 are not equal, f is not differentiable at x = 1

Question 10:

Prove that the greatest integer function defined by $f(x) = [x], 0 < x < 3$ is not differentiable at x = 1 and x = 2.

Answer

The given function f is $f(x) = [x], 0 < x < 3$

It is known that a function f is differentiable at a point x = c in its domain if both

$\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h}$ and $\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at x = 1, consider the left hand limit of f at x = 1

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{[1+h]-[1]}{h}$$

$$=\lim_{h\to 0^-}\frac{0-1}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

Consider the right hand limit of f at $x=1$

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{[1+h]-[1]}{h}$$

$$=\lim_{h\to 0^+}\frac{1-1}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at x = 1 are not equal, f is not differentiable at x = 1

To check the differentiability of the given function at x = 2, consider the left hand limit of f at x = 2

Since the left and right hand limits of f at x = 2 are not equal, f is not differentiable at x = 2

Exercise 5.3

Question 1:

Find $\frac{dy}{dx}$: $2x+3y=\sin x$

Answer

The given relationship is $2x+3y=\sin x$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(2x+3y)=\frac{d}{dx}(\sin x)$$
$$\Rightarrow \frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\cos x$$
$$\Rightarrow 2+3\frac{dy}{dx}=\cos x$$
$$\Rightarrow 3\frac{dy}{dx}=\cos x-2$$
$$\therefore \frac{dy}{dx}=\frac{\cos x-2}{3}$$

Question 2:

Find $\frac{dy}{dx}$

Answer

The given relationship is $2x+3y=\sin y$

Differentiating this relationship with respect to x, we obtain

$$\Rightarrow 2+3\frac{dy}{dx}=\cos y\frac{dy}{dx} \qquad \text{[By using chain rule]}$$

$$\Rightarrow 2=(\cos y-3)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{2}{\cos y-3}$$

Question 3:

Find $\frac{dy}{dx}$

$$ax+by^2=\cos y$$

Answer

The given relationship is $ax+by^2=\cos y$

Differentiating this relationship with respect to x, we obtain

Using chain rule, we obtain $\frac{d}{dx}(y^2)=2y\frac{dy}{dx}$ and $\frac{d}{dx}(\cos y)=-\sin y\frac{dy}{dx}$ ...(2)

From (1) and (2), we obtain

Question 4:

Find $\frac{dy}{dx}$

Answer

The given relationship is $xy + y^2 = \tan x + y$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(xy + y^2\right) = \frac{d}{dx}\left(\tan x + y\right)$$

$$\Rightarrow \frac{d}{dx}(xy) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(\tan x) + \frac{dy}{dx}$$

$$\Rightarrow \left[y \cdot \frac{d}{dx}(x) + x.\frac{dy}{dx}\right] + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx} \qquad \text{[Using product rule and chain rule]}$$

$$\Rightarrow y \cdot 1 + x.\frac{dy}{dx} + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow (x + 2y - 1)\frac{dy}{dx} = \sec^2 x - y$$

$$\therefore \frac{dy}{dx} = \frac{\sec^2 x - y}{(x + 2y - 1)}$$

Question 5:

Find $\frac{dy}{dx}$

Answer

The given relationship is $x^2 + xy + y^2 = 100$

Differentiating this relationship with respect to x, we obtain

$$\Rightarrow 2x+\left[y\cdot\frac{d}{dx}(x)+x\cdot\frac{dy}{dx}\right]+2y\frac{dy}{dx}=0 \qquad [\text{Using product rule and chain rule}]$$

$$\Rightarrow 2x+y\cdot 1+x\cdot\frac{dy}{dx}+2y\frac{dy}{dx}=0$$

$$\frac{d}{dx}\left(\sin^2 y+\cos xy\right)=\frac{d}{dx}(\pi)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 y\right)+\frac{d}{dx}\left(\cos xy\right)=0 \qquad \ldots(1)$$

Question 6:

Find

$$\frac{d}{dx}\left(\sin^2 y\right)=2\sin y\frac{d}{dx}(\sin y)=2\sin y\cos y\frac{dy}{dx} \qquad \ldots(2)$$

$$\frac{d}{dx}(\cos xy)=-\sin xy\frac{d}{dx}(xy)=-\sin xy\left[y\frac{d}{dx}(x)+x\frac{dy}{dx}\right]$$

Answer

The given relationship is

$$=-\sin xy\left[y.1+x\frac{dy}{dx}\right]=-y\sin xy-x\sin xy\frac{dy}{dx} \qquad \ldots(3)$$

Differentiating this relationship with respect to x, we o

Question 7:

$$2\sin y\cos y\frac{dy}{dx}-y\sin xy-x\sin xy\frac{dy}{dx}=0$$

Find

$$\Rightarrow (2\sin y\cos y-x\sin xy)\frac{dy}{dx}=y\sin xy$$

$$\Rightarrow (\sin 2y-x\sin xy)\frac{dy}{dx}=y\sin xy$$

$$\therefore \frac{dy}{dx}=\frac{y\sin xy}{\sin 2y-x\sin xy}$$

Answer

The given relationship is $\sin^2 y+\cos xy=\pi$

Differentiating this relationship with respect to x, we obtain

$\sin^2 x+\cos^2 y=1$

Using chain rule, we obtain

From (1), (2), and (3), we obtain

Question 8:

Find $\frac{dy}{dx}$

Answer

The given relationship is $\sin^2 x + \cos^2 y = 1$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}\left(\sin^2 x+\cos^2 y\right)=\frac{d}{dx}(1)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 x\right)+\frac{d}{dx}\left(\cos^2 y\right)=0$$

$$\Rightarrow 2\sin x\cdot\frac{d}{dx}(\sin x)+2\cos y\cdot\frac{d}{dx}(\cos y)=0$$

$$\Rightarrow 2\sin x\cos x+2\cos y(-\sin y)\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \sin 2x-\sin 2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{\sin 2x}{\sin 2y}$$

Question 9:

Find $\frac{dy}{dx}$

Answer $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

The given relationship is $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \sin y=\frac{2x}{1+x^2}$$

Differentiating this relationship with respect to x, we obtain

The function, $\frac{2x}{1+x^2}$, is of the form of $\frac{u}{v}$.

Therefore, by quotient rule, we obtain

$$\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)=\frac{(1+x^2)\cdot\frac{d}{dx}(2x)-2x\cdot\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$=\frac{(1+x^2)\cdot 2-2x\cdot[0+2x]}{(1+x^2)^2}=\frac{2+2x^2-4x^2}{(1+x^2)^2}=\frac{2(1-x^2)}{(1+x^2)^2} \quad ...(2)$$

Also, $\sin y=\frac{2x}{1+x^2}$

$$\Rightarrow \cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}=\sqrt{\frac{(1+x^2)^2-4x^2}{(1+x^2)^2}}$$

$$=\sqrt{\frac{(1-x^2)^2}{(1+x^2)^2}}=\frac{1-x^2}{1+x^2} \quad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{1-x^2}{1+x^2}\times\frac{dy}{dx}=\frac{2(1-x^2)}{(1+x^2)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{1+x^2}$$

Question 10:

Find $\frac{dy}{dx}$

Answer

The given relationship is $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

$$y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\Rightarrow \tan y=\frac{3x-x^3}{1-3x^2}$$

It is known that, $\tan y=\dfrac{3\tan\frac{y}{3}-\tan^3\frac{y}{3}}{1-3\tan^2\frac{y}{3}}$ ...(2)

Comparing equations (1) and (2), we obtain

$$x=\tan\frac{y}{3}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x)=\frac{d}{dx}\left(\tan\frac{y}{3}\right)$$

$$\Rightarrow 1=\sec^2\frac{y}{3}\cdot\frac{d}{dx}\left(\frac{y}{3}\right)$$

$$\Rightarrow 1=\sec^2\frac{y}{3}\cdot\frac{1}{3}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{3}{\sec^2\frac{y}{3}}=\frac{3}{1+\tan^2\frac{y}{3}}$$

$$\therefore \frac{dy}{dx}=\frac{3}{1+x^2}$$

Question 11:

Find $\dfrac{dy}{dx}$

Answer

The given relationship is,

$$y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \cos y=\frac{1-x^2}{1+x^2}$$

$$\Rightarrow \frac{1-\tan^2\frac{y}{2}}{1+\tan^2\frac{y}{2}}=\frac{1-x^2}{1+x^2}$$

On comparing L.H.S. and R.H.S. of the above relationship, we obtain

$$\tan\frac{y}{2}=x$$

Differentiating this relationship with respect to x, we obtain

$$\sec^2\frac{y}{2}\cdot\frac{d}{dx}\left(\frac{y}{2}\right)=\frac{d}{dx}(x)$$

$$\Rightarrow \sec^2\frac{y}{2}\times\frac{1}{2}\frac{dy}{dx}=1$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{\sec^2\frac{y}{2}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2}{1+\tan^2\frac{y}{2}}$$

$$\therefore \frac{dy}{dx}=\frac{1}{1+x^2}$$

Question 12:

Find $\frac{dy}{dx}$

Answer

The given relationship is $y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Differentiating this relationship with respect to x, we obtain

$$\Rightarrow \sin y=\frac{1-x^2}{1+x^2}$$

$$\frac{d}{dx}(\sin y)=\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) \qquad ...(1)$$

Using chain rule, we obtain

$$\frac{d}{dx}(\sin y)=\cos y\cdot\frac{dy}{dx}$$

$$\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}$$

$$=\sqrt{\frac{\left(1+x^2\right)^2-\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}=\sqrt{\frac{4x^2}{\left(1+x^2\right)^2}}=\frac{2x}{1+x^2}$$

$$\therefore \frac{d}{dx}(\sin y)=\frac{2x}{1+x^2}\frac{dy}{dx} \qquad ...(2)$$

From (1), (2), and (3), we obtain

$$\frac{2x}{1+x^2}\frac{dy}{dx}=\frac{-4x}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{1+x^2}$$

Alternate method

$$y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$\Rightarrow$

$$\sin y=\frac{1-x^2}{}$$

$$\Rightarrow \left(1+x^2\right)\sin y=1-x^2$$

$$\Rightarrow \left(1+\sin y\right)x^2=1-\sin y$$

$$\Rightarrow x^2=\frac{1-\sin y}{1+\sin y}$$

$$\Rightarrow x^2=\frac{\left(\cos\frac{y}{2}-\sin\frac{y}{2}\right)^2}{\left(\cos\frac{y}{2}+\sin\frac{y}{2}\right)^2}$$

$$\Rightarrow x=\frac{\cos\frac{y}{2}-\sin\frac{y}{2}}{\cos\frac{y}{2}+\sin\frac{y}{2}}$$

$$\Rightarrow x=\frac{1-\tan\frac{y}{2}}{1+\tan\frac{y}{2}}$$

$$\Rightarrow x=\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)$$

Differentiating this relationship with respect to x, we obtain

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$$\Rightarrow -\sqrt{1-\cos^2 y}\,\frac{dy}{dx}=\frac{\left(1+x^2\right)\times 2-2x\cdot 2x}{\left(1+x^2\right)^2}$$

$$\Rightarrow \left[\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}\right]\frac{dy}{dx}=-\left[\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}\right]$$

$$\frac{\pi}{4}-\frac{y}{2}\Bigg)$$

$$\cdot\left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow \sqrt{\frac{\left(1+x^2\right)^2-4x^2}{\left(1+x^2\right)^2}}\,\frac{dy}{dx}=\frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \sqrt{\frac{\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}\,\frac{dy}{dx}=\frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{1-x^2}{1+x^2}\cdot\frac{dy}{dx}=\frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{1+x^2}$$

Question 13:

Find $\frac{dy}{dx}$

Answer

The given relationship is $y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$

Differentiating this relationship with respect to x, we obtain

Question 14:

Find

Answer

The given relationship is $y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

Differentiating this relationship with respect to x, we obtain

$$\cos y\frac{dy}{dx}=2\left[x\frac{d}{dx}\left(\sqrt{1-x^2}\right)+\sqrt{1-x^2}\frac{dx}{dx}\right]$$

$$\Rightarrow\sqrt{1-\sin^2 y}\frac{dy}{dx}=2\left[\frac{x}{2}.\frac{-2x}{\sqrt{1-x^2}}+\sqrt{1-x^2}\right]$$

$$\Rightarrow\sqrt{1-\left(2x\sqrt{1-x^2}\right)^2}\frac{dy}{dx}=2\left[\frac{-x^2+1-x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\sqrt{1-4x^2\left(1-x^2\right)}\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\sqrt{\left(1-2x^2\right)^2}\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\left(1-2x^2\right)\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\frac{dy}{dx}=\frac{2}{\sqrt{1-x^2}}$$

Question 15:

Find $\frac{dy}{dx}$

Answer

The given relationship is $y = \sec^{-1}\left(\frac{1}{2x^2 - 1}\right)$

$$\Rightarrow \sec y = \frac{1}{2x^2-1}$$

$$\Rightarrow \cos y = 2x^2 - 1$$

$$\Rightarrow 2x^2 = 1 + \cos y$$

$$\Rightarrow 2x^2 = 2\cos^2\frac{y}{2}$$

$$\Rightarrow x = \cos\frac{y}{2}$$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\cos\frac{y}{2}\right)$$

$$\Rightarrow 1 = -\sin\frac{y}{2}\cdot\frac{d}{dx}\left(\frac{y}{2}\right)$$

$$\Rightarrow \frac{-1}{\sin\frac{y}{2}} = \frac{1}{2}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sin\frac{y}{2}} = \frac{-2}{\sqrt{1-\cos^2\frac{y}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1-x^2}}$$

Question 1:

Differentiate the following w.r.t. x:

$$\frac{e^x}{\sin x}$$

Answer

Let $y = \dfrac{e^x}{\sin x}$

By using the quotient rule, we obtain

Question 2:

Differentiate the following w.r.t. x:

Answer

Let $y = e^{\sin^{-1} x}$

By using the chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{\sin^{-1}x}\right)$$

$$\Rightarrow \frac{dy}{dx}=e^{\sin^{-1}x}\cdot\frac{d}{dx}\left(\sin^{-1}x\right)$$

$$=e^{\sin^{-1}x}\cdot\frac{1}{\sqrt{1-x^2}}$$

$$=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx}=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}, x\in(-1,1)$$

Question 2:

Show that the function given by f(x) = e^{2x} is strictly increasing on R.

Answer

Let x_1 and x_2 be any two numbers in R.

Then, we have:

Hence, f is strictly increasing on R.

Question 3:

Differentiate the following w.r.t. x:

Answer

Let $y=e^{x^3}$

By using the chain rule, we obtain

Question 4:

Differentiate the following w.r.t. x:

Answer

Let $y = \sin\left(\tan^{-1} e^{-x}\right)$

By using the chain rule, we obtain

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left[\sin\left(\tan^{-1} e^{-x}\right)\right] \\
&= \cos\left(\tan^{-1} e^{-x}\right)\cdot\frac{d}{dx}\left(\tan^{-1} e^{-x}\right) \\
&= \cos\left(\tan^{-1} e^{-x}\right)\cdot\frac{1}{1+\left(e^{-x}\right)^2}\cdot\frac{d}{dx}\left(e^{-x}\right) \\
&= \frac{\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}\cdot e^{-x}\cdot\frac{d}{dx}(-x) \\
&= \frac{e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}\times(-1) \\
&= \frac{-e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}
\end{aligned}$$

Question 5:

Differentiate the following w.r.t. x:

Answer

Let $y = \log\left(\cos e^{x}\right)$

By using the chain rule, we obtain

Question 6:

Differentiate the following w.r.t. x:

$e^{x}+e^{x^{2}}+\ldots+e^{x^{5}}$

Answer

$$\frac{d}{dx}\left(e^{x}+e^{x^{2}}+\ldots+e^{x^{5}}\right)$$
$$=\frac{d}{dx}\left(e^{x}\right)+\frac{d}{dx}\left(e^{x^{2}}\right)+\frac{d}{dx}\left(e^{x^{3}}\right)+\frac{d}{dx}\left(e^{x^{4}}\right)+\frac{d}{dx}\left(e^{x^{5}}\right)$$
$$=e^{x}+\left[e^{x^{2}}\times\frac{d}{dx}\left(x^{2}\right)\right]+\left[e^{x^{3}}\cdot\frac{d}{dx}\left(x^{3}\right)\right]+\left[e^{x^{4}}\cdot\frac{d}{dx}\left(x^{4}\right)\right]+\left[e^{x^{5}}\cdot\frac{d}{dx}\left(x^{5}\right)\right]$$
$$=e^{x}+\left(e^{x^{2}}\times 2x\right)+\left(e^{x^{3}}\times 3x^{2}\right)+\left(e^{x^{4}}\times 4x^{3}\right)+\left(e^{x^{5}}\times 5x^{4}\right)$$
$$=e^{x}+2xe^{x^{2}}+3x^{2}e^{x^{3}}+4x^{3}e^{x^{4}}+5x^{4}e^{x^{5}}$$

Question 7:

Differentiate the following w.r.t. x:

Answer

Let $y=\sqrt{e^{\sqrt{x}}}$

Then, $y^{2}=e^{\sqrt{x}}$

By differentiating this relationship with respect to x, we obtain

$$y^2 = e^{\sqrt{x}}$$

$$\Rightarrow 2y\frac{dy}{dx} = e^{\sqrt{x}}\frac{d}{dx}\left(\sqrt{x}\right) \qquad \text{[By applying the chain rule]}$$

$$\Rightarrow 2y\frac{dy}{dx} = e^{\sqrt{x}}\frac{1}{2}\cdot\frac{1}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4y\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\,\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}, \; x > 0$$

Question 8:

Differentiate the following w.r.t. x:

Answer

Let $y = \log(\log x)$

By using the chain rule, we obtain

$$= \frac{1}{x\log x}, \; x > 1$$

Question 9:

Differentiate the following w.r.t. x:

Answer

Let $y=\dfrac{\cos x}{\log x}$

By using the quotient rule, we obtain

$$\frac{dy}{dx}=\frac{\frac{d}{dx}(\cos x)\times\log x-\cos x\times\frac{d}{dx}(\log x)}{(\log x)^2}$$

$$=\frac{-\sin x\log x-\cos x\times\frac{1}{x}}{(\log x)^2}$$

$$=\frac{-[x\log x.\sin x+\cos x]}{x(\log x)^2},x>0$$

Question 10:

Differentiate the following w.r.t. x:

Answer

Let $y=\cos\left(\log x+e^x\right)$

By using the chain rule, we obtain

Exercise 5.5

Question 1:

Differentiate the function with respect to x.

$\cos x.\cos 2x.\cos 3x$

Answer

Let $y = \cos x.\cos 2x.\cos 3x$

Taking logarithm on both the sides, we obtain

$$\log y = \log(\cos x.\cos 2x.\cos 3x)$$
$$\Rightarrow \log y = \log(\cos x) + \log(\cos 2x) + \log(\cos 3x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\cos x}\cdot\frac{d}{dx}(\cos x) + \frac{1}{\cos 2x}\cdot\frac{d}{dx}(\cos 2x) + \frac{1}{\cos 3x}\cdot\frac{d}{dx}(\cos 3x)$$
$$\Rightarrow \frac{dy}{dx} = y\left[-\frac{\sin x}{\cos x} - \frac{\sin 2x}{\cos 2x}\cdot\frac{d}{dx}(2x) - \frac{\sin 3x}{\cos 3x}\cdot\frac{d}{dx}(3x)\right]$$
$$\therefore \frac{dy}{dx} = -\cos x.\cos 2x.\cos 3x[\tan x + 2\tan 2x + 3\tan 3x]$$

Question 2:

Differentiate the function with respect to x.

Answer

Taking logarithm on both the sides, we obtain

$$\log y=\log\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

$$\Rightarrow \log y=\frac{1}{2}\log\left[\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right]$$

$$\Rightarrow \log y=\frac{1}{2}\left[\log\{(x-1)(x-2)\}-\log\{(x-3)(x-4)(x-5)\}\right]$$

$$\Rightarrow \log y=\frac{1}{2}\left[\log(x-1)+\log(x-2)-\log(x-3)-\log(x-4)-\log(x-5)\right]$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\left[\begin{array}{l}\frac{1}{x-1}\cdot\frac{d}{dx}(x-1)+\frac{1}{x-2}\cdot\frac{d}{dx}(x-2)-\frac{1}{x-3}\cdot\frac{d}{dx}(x-3)\\ -\frac{1}{x-4}\cdot\frac{d}{dx}(x-4)-\frac{1}{x-5}\cdot\frac{d}{dx}(x-5)\end{array}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{2}\left(\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right)$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right]$$

Question 3:

$(\log x)^{\cos x}$

Differentiate the function with respect to x.

$\text{Let } y=(\log x)^{\cos x}$

Answer

$\log y=\cos x\cdot\log(\log x)$

Taking logarithm on both the sides, we obtain

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d}{dx}(\cos x)\times\log(\log x)+\cos x\times\frac{d}{dx}[\log(\log x)]$$

$$\Rightarrow\frac{1}{y}\cdot\frac{dy}{dx}=-\sin x\log(\log x)+\cos x\times\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{dy}{dx}=y\left[-\sin x\log(\log x)+\frac{\cos x}{\log x}\times\frac{1}{x}\right]$$

$$\therefore\frac{dy}{dx}=(\log x)^{\cos x}\left[\frac{\cos x}{x\log x}-\sin x\log(\log x)\right]$$

Question 4:

Differentiate the function with respect to x.

Answer

u = x^x

Taking logarithm on both the sides, we obtain

Differentiating both sides with respect to x, we obtain

v = $2^{\sin x}$

Taking logarithm on both the sides with respect to x, we obtain

$\log v = \sin x \cdot \log 2$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log 2 \cdot \frac{d}{dx}(\sin x)$$
$$\Rightarrow \frac{dv}{dx} = v \log 2 \cos x$$
$$\Rightarrow \frac{dv}{dx} = 2^{\sin x} \cos x \log 2$$
$$\therefore \frac{dy}{dx} = x^x (1 + \log x) - 2^{\sin x} \cos x \log 2$$

Question 5:

$(x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$

Differentiate the function with respect to x.

Answer

Let $y = (x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$

Taking logarithm on both the sides, we obtain

$$\log y = \log(x+3)^2 + \log(x+4)^3 + \log(x+5)^4$$
$$\Rightarrow \log y = 2\log(x+3) + 3\log(x+4) + 4\log(x+5)$$

Differentiating both sides with respect to x, we obtain

Question 6:

Differentiate the function with respect to x.

$$\left(x+\frac{1}{x}\right)^{x}+x^{\left(1+\frac{1}{x}\right)}$$

Answer

$$\text{Let } y=\left(x+\frac{1}{x}\right)^{x}+x^{\left(1+\frac{1}{x}\right)}$$

$$\text{Also, let } u=\left(x+\frac{1}{x}\right)^{x} \text{ and } v=x^{\left(1+\frac{1}{x}\right)}$$

$$\therefore y=u+v$$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad \ldots(1)$$

Differentiating both sides with respect to x, we obtain

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$$\frac{1}{u}\cdot\frac{du}{dx}=\frac{d}{dx}(x)\times\log\left(x+\frac{1}{x}\right)+x\times\frac{d}{dx}\left[\log\left(x+\frac{1}{x}\right)\right]$$

$$\Rightarrow\frac{1}{u}\frac{du}{dx}=1\times\log\left(x+\frac{1}{x}\right)+x\times\frac{1}{\left(x+\frac{1}{x}\right)}\cdot\frac{d}{dx}\left(x+\frac{1}{x}\right)$$

$$\Rightarrow\frac{du}{dx}=u\left[\log\left(x+\frac{1}{x}\right)+\frac{x}{\left(x+\frac{1}{x}\right)}\times\left(1-\frac{1}{x^2}\right)\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right)+\frac{\left(x-\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)}\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right)+\frac{x^2-1}{x^2+1}\right]$$

$$\Rightarrow\frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right] \quad ...(2)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\left[\frac{d}{dx}\left(1+\frac{1}{x}\right)\right]\times\log x+\left(1+\frac{1}{x}\right)\cdot\frac{d}{dx}\log x$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=\left(-\frac{1}{x^2}\right)\log x+\left(1+\frac{1}{x}\right)\cdot\frac{1}{x}$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=-\frac{\log x}{x^2}+\frac{1}{x}+\frac{1}{x^2}$$

$$\Rightarrow\frac{dv}{dx}=v\left[\frac{-\log x+x+1}{x^2}\right]$$

$$\Rightarrow\frac{dv}{dx}=x^{\left(1+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^2}\right) \quad ...(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\frac{dy}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right]+x^{\left(1+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^2}\right)$$

Question 7:

$(\log x)^x + x^{\log x}$

Differentiate the function with respect to x.

Answer

$u = (\log x)^x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\times\log(\log x)+x\cdot\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow\frac{du}{dx}=u\left[1\times\log(\log x)+x\cdot\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x}\left[\log(\log x)+\frac{x}{\log x}\cdot\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x}\left[\log(\log x)+\frac{1}{\log x}\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x}\left[\frac{\log(\log x)\cdot\log x+1}{\log x}\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x-1}\left[1+\log x.\log(\log x)\right]\qquad ...(2)$$

$$v=x^{\log x}$$

$$\Rightarrow\log v=\log\left(x^{\log x}\right)$$

$$\Rightarrow\log v=\log x\log x=(\log x)^{2}$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\frac{d}{dx}\left[(\log x)^{2}\right]$$

$$\Rightarrow\frac{1}{v}.\frac{dv}{dx}=2(\log x)\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{dv}{dx}=2v(\log x)\cdot\frac{1}{x}$$

$$\Rightarrow\frac{dv}{dx}=2x^{\log x}\frac{\log x}{x}$$

$$\Rightarrow\frac{dv}{dx}=2x^{\log x-1}\cdot\log x\qquad ...(3)$$

$$\frac{dy}{dx}=(\log x)^{x-1}\left[1+\log x.\log(\log x)\right]+2x^{\log x-1}\cdot\log x$$

Therefore, from (1), (2), and (3), we obtain

Question 8:

Differentiate the function with respect to x.

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Answer

$(\sin x)^x + \sin^{-1}\sqrt{x}$

Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

Also, let $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ...(1)$$

$u = (\sin x)^x$

$\Rightarrow \log u = \log(\sin x)^x$

$\Rightarrow \log u = x\log(\sin x)$

Differentiating both sides with respect to x, we obtain

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x)\times\log(\sin x) + x\times\frac{d}{dx}\left[\log(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx} = u\left[1\cdot\log(\sin x) + x\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x\left[\log(\sin x) + \frac{x}{\sin x}\cdot\cos x\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x(x\cot x + \log\sin x) \quad ...(2)$$

Differentiating both sides with respect to x, we obtain

Therefore, from (1), (2), and (3), we obtain

Question 9:

Differentiate the function with respect to x.

Answer

$$x^{\sin x}+(\sin x)^{\cos x}$$

Let $y=x^{\sin x}+(\sin x)^{\cos x}$

Also, let $u=x^{\sin x}$ and $v=(\sin x)^{\cos x}$

$\therefore y=u+v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ...(1)$$

$u=x^{\sin x}$

$\Rightarrow \log u=\log\left(x^{\sin x}\right)$

$\Rightarrow \log u=\sin x\log x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(\sin x)\cdot\log x+\sin x\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx}=u\left[\cos x\log x+\sin x\cdot\frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx}=x^{\sin x}\left[\cos x\log x+\frac{\sin x}{x}\right] \qquad ...(2)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}(\cos x)\times\log(\sin x) + \cos x \times \frac{d}{dx}\left[\log(\sin x)\right]$$

$$\Rightarrow \frac{dv}{dx} = v\left[-\sin x.\log(\sin x) + \cos x\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\left[-\sin x\log\sin x + \frac{\cos x}{\sin x}\cos x\right]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\left[-\sin x\log\sin x + \cot x\cos x\right]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\left[\cot x\cos x - \sin x\log\sin x\right] \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx} = x^{\sin x}\left(\cos x\log x + \frac{\sin x}{x}\right) + (\sin x)^{\cos x}\left[\cos x\cot x - \sin x\log\sin x\right]$$

Question 10:

Differentiate the function with respect to x.

$$x^{x\cos x} + \frac{x^2+1}{x^2-1}$$

Answer

$$\text{Let } y = x^{x\cos x} + \frac{x^2+1}{x^2-1}$$

$$\text{Also, let } u = x^{x\cos x} \text{ and } v = \frac{x^2+1}{x^2-1}$$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(1)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x)\cdot\cos x\cdot\log x+x\cdot\frac{d}{dx}(\cos x)\cdot\log x+x\cos x\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{du}{dx}=u\left[1\cdot\cos x\cdot\log x+x\cdot(-\sin x)\log x+x\cos x\cdot\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=x^{x\cos x}(\cos x\log x-x\sin x\log x+\cos x)$$

$$\Rightarrow\frac{du}{dx}=x^{x\cos x}\left[\cos x(1+\log x)-x\sin x\log x\right]\qquad ...(2)$$

$$v=\frac{x^2+1}{x^2-1}$$

$$\Rightarrow\log v=\log(x^2+1)-\log(x^2-1)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\frac{2x}{x^2+1}-\frac{2x}{x^2-1}$$

$$\Rightarrow\frac{dv}{dx}=v\left[\frac{2x(x^2-1)-2x(x^2+1)}{(x^2+1)(x^2-1)}\right]$$

$$\Rightarrow\frac{dv}{dx}=\frac{x^2+1}{x^2-1}\times\left[\frac{-4x}{(x^2+1)(x^2-1)}\right]$$

$$\Rightarrow\frac{dv}{dx}=\frac{-4x}{(x^2-1)^2}\qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx}=x^{x\cos x}\left[\cos x(1+\log x)-x\sin x\log x\right]-\frac{4x}{(x^2-1)^2}$$

Question 11:

Differentiate the function with respect to x.

$$(x\cos x)^x+(x\sin x)^{\frac{1}{x}}$$

Answer

$$\text{Let } y=(x\cos x)^x+(x\sin x)^{\frac{1}{x}}$$

$$\text{Also, let } u=(x\cos x)^x \text{ and } v=(x\sin x)^{\frac{1}{x}}$$

$$\therefore y=u+v$$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ...(1)$$

$$u=(x\cos x)^x$$

$$\Rightarrow \log u=\log(x\cos x)^x$$

$$\Rightarrow \log u=x\log(x\cos x)$$

$$\Rightarrow \log u=x[\log x+\log\cos x]$$

$$\Rightarrow \log u=x\log x+x\log\cos x$$

Differentiating both sides with respect to x, we obtain

$$v=(x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v=\log(x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v=\frac{1}{x}\log(x\sin x)$$

$$\Rightarrow \log v=\frac{1}{x}(\log x+\log\sin x)$$

$$\Rightarrow \log v=\frac{1}{x}\log x+\frac{1}{x}\log\sin x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{x}\log x\right)+\frac{d}{dx}\left[\frac{1}{x}\log(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\left[\log x\cdot\frac{d}{dx}\left(\frac{1}{x}\right)+\frac{1}{x}\cdot\frac{d}{dx}(\log x)\right]+\left[\log(\sin x)\cdot\frac{d}{dx}\left(\frac{1}{x}\right)+\frac{1}{x}\cdot\frac{d}{dx}\{\log(\sin x)\}\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\left[\log x\cdot\left(-\frac{1}{x^2}\right)+\frac{1}{x}\cdot\frac{1}{x}\right]+\left[\log(\sin x)\cdot\left(-\frac{1}{x^2}\right)+\frac{1}{x}\cdot\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\frac{1}{x^2}(1-\log x)+\left[-\frac{\log(\sin x)}{x^2}+\frac{1}{x\sin x}\cdot\cos x\right]$$

$$\Rightarrow \frac{dv}{dx}=(x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x}{x^2}+\frac{-\log(\sin x)+x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx}=(x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x-\log(\sin x)+x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx}=(x\sin x)^{\frac{1}{x}}\left[\frac{1-\log(x\sin x)+x\cot x}{x^2}\right] \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

Question 12:

Find $\frac{dy}{dx}$ of function.

Answer

The given function is $x^y + y^x = 1$

$$x^y + y^x = 1$$

Let $x^y = u$ and $y^x = v$

Then, the function becomes u + v = 1

$$\therefore \frac{du}{dx} + \frac{dv}{dx} = 0 \qquad \dots(1)$$

$$u = x^y$$

$$\Rightarrow \log u = \log\left(x^y\right)$$

$$\Rightarrow \log u = y \log x$$

Differentiating both sides with respect to x, we obtain

Differentiating both sides with respect to x, we obtain

From (1), (2), and (3), we obtain

$$x^y\left(\log x\frac{dy}{dx}+\frac{y}{x}\right)+y^x\left(\log y+\frac{x}{y}\frac{dy}{dx}\right)=0$$

$$\Rightarrow\left(x^y\log x+xy^{x-1}\right)\frac{dy}{dx}=-\left(yx^{y-1}+y^x\log y\right)$$

$$\therefore\frac{dy}{dx}=-\frac{yx^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}$$

Question 13:

Find $\frac{dy}{dx}$ of function.

$y^x=x^y$

Answer

The given function is $y^x=x^y$

Taking logarithm on both the sides, we obtain

Differentiating both sides with respect to x, we obtain

$$x\log y=y\log x$$

$$\log y\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log y)=\log x\cdot\frac{d}{dx}(y)+y\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\log y\cdot 1+x\cdot\frac{1}{y}\cdot\frac{dy}{dx}=\log x\cdot\frac{dy}{dx}+y\cdot\frac{1}{x}$$

$$\Rightarrow\log y+\frac{x}{y}\frac{dy}{dx}=\log x\frac{dy}{dx}+\frac{y}{x}$$

$$\Rightarrow\left(\frac{x}{y}-\log x\right)\frac{dy}{dx}=\frac{y}{x}-\log y$$

$$\Rightarrow\left(\frac{x-y\log x}{y}\right)\frac{dy}{dx}=\frac{y-x\log y}{x}$$

$$\therefore\frac{dy}{dx}=\frac{y}{x}\left(\frac{y-x\log y}{x-y\log x}\right)$$

Question 14:

Find $\frac{dy}{dx}$ of function.

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Answer

$(\cos x)^y = (\cos y)^x$

The given function is $(\cos x)^y = (\cos y)^x$

Taking logarithm on both the sides, we obtain

Differentiating both sides, we obtain

$y \log \cos x = x \log \cos y$

$$\log \cos x \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log \cos x) = \log \cos y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log \cos y)$$

$$\Rightarrow \log \cos x \frac{dy}{dx} + y \cdot \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x) = \log \cos y \cdot 1 + x \cdot \frac{1}{\cos y} \cdot \frac{d}{dx}(\cos y)$$

$$\Rightarrow \log \cos x \frac{dy}{dx} + \frac{y}{\cos x} \cdot (-\sin x) = \log \cos y + \frac{x}{\cos y}(-\sin y) \cdot \frac{dy}{dx}$$

$$\Rightarrow \log \cos x \frac{dy}{dx} - y \tan x = \log \cos y - x \tan y \frac{dy}{dx}$$

$$\Rightarrow (\log \cos x + x \tan y)\frac{dy}{dx} = y \tan x + \log \cos y$$

$$\therefore \frac{dy}{dx} = \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x}$$

Question 15:

Find $\frac{dy}{dx}$ of function.

$xy = e^{(x-y)}$

Answer

The given function is $xy = e^{(x-y)}$

Taking logarithm on both the sides, we obtain

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}(\log x)+\frac{d}{dx}(\log y)=\frac{d}{dx}(x)-\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=1-\frac{dy}{dx}$$

$$\Rightarrow \left(1+\frac{1}{y}\right)\frac{dy}{dx}=1-\frac{1}{x}$$

$$\Rightarrow \left(\frac{y+1}{y}\right)\frac{dy}{dx}=\frac{x-1}{x}$$

$$\therefore \frac{dy}{dx}=\frac{y(x-1)}{x(y+1)}$$

Question 16:

Find the derivative of the function given by $f(x)=(1+x)\left(1+x^2\right)\left(1+x^4\right)\left(1+x^8\right)$ and hence find $f'(1)$.

Answer

The given relationship is $f(x)=(1+x)\left(1+x^2\right)\left(1+x^4\right)\left(1+x^8\right)$

Taking logarithm on both the sides, we obtain

Differentiating both sides with respect to x, we obtain

$$\frac{1}{f(x)}\cdot\frac{d}{dx}[f(x)]=\frac{d}{dx}\log(1+x)+\frac{d}{dx}\log(1+x^2)+\frac{d}{dx}\log(1+x^4)+\frac{d}{dx}\log(1+x^8)$$

$$\Rightarrow\frac{1}{f(x)}\cdot f'(x)=\frac{1}{1+x}\cdot\frac{d}{dx}(1+x)+\frac{1}{1+x^2}\cdot\frac{d}{dx}(1+x^2)+\frac{1}{1+x^4}\cdot\frac{d}{dx}(1+x^4)+\frac{1}{1+x^8}\cdot\frac{d}{dx}(1+x^8)$$

$$\Rightarrow f'(x)=f(x)\left[\frac{1}{1+x}+\frac{1}{1+x^2}.2x+\frac{1}{1+x^4}\cdot 4x^3+\frac{1}{1+x^8}\cdot 8x^7\right]$$

$$\therefore f'(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}\right]$$

Hence, $f'(1)=(1+1)(1+1^2)(1+1^4)(1+1^8)\left[\frac{1}{1+1}+\frac{2\times1}{1+1^2}+\frac{4\times1^3}{1+1^4}+\frac{8\times1^7}{1+1^8}\right]$

$$=2\times2\times2\times2\left[\frac{1}{2}+\frac{2}{2}+\frac{4}{2}+\frac{8}{2}\right]$$

$$=16\times\left(\frac{1+2+4+8}{2}\right)$$

$$=16\times\frac{15}{2}=120$$

Question 17:

Differentiate $(x^5-5x+8)(x^3+7x+9)$ in three ways mentioned below

(i) By using product rule.

(ii) By expanding the product to obtain a single polynomial.

(iii By logarithmic differentiation.

Do they all give the same answer?

Answer

(i) Let $y=(x^5-5x+8)(x^3+7x+9)$

Let $x^2 - 5x + 8 = u$ and $x^3 + 7x + 9 = v$

$\therefore y = uv$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$ (By using product rule)

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^2 - 5x + 8) \cdot (x^3 + 7x + 9) + (x^2 - 5x + 8) \cdot \frac{d}{dx}(x^3 + 7x + 9)$

$\Rightarrow \frac{dy}{dx} = (2x - 5)(x^3 + 7x + 9) + (x^2 - 5x + 8)(3x^2 + 7)$

$\Rightarrow \frac{dy}{dx} = 2x(x^3 + 7x + 9) - 5(x^3 + 7x + 9) + x^2(3x^2 + 7) - 5x(3x^2 + 7) + 8(3x^2 + 7)$

$\Rightarrow \frac{dy}{dx} = (2x^4 + 14x^2 + 18x) - 5x^3 - 35x - 45 + (3x^4 + 7x^2) - 15x^3 - 35x + 24x^2 + 56$

(ii) $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$

$= x^2(x^3 + 7x + 9) - 5x(x^3 + 7x + 9) + 8(x^3 + 7x + 9)$

$= x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^2 - 45x + 8x^3 + 56x + 72$

$= x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72$

$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72)$

$= \frac{d}{dx}(x^5) - 5\frac{d}{dx}(x^4) + 15\frac{d}{dx}(x^3) - 26\frac{d}{dx}(x^2) + 11\frac{d}{dx}(x) + \frac{d}{dx}(72)$

$= 5x^4 - 5 \times 4x^3 + 15 \times 3x^2 - 26 \times 2x + 11 \times 1 + 0$

$= 5x^4 - 20x^3 + 45x^2 - 52x + 11$

(iii) $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$

Taking logarithm on both the sides, we obtain

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\log\left(x^2-5x+8\right)+\frac{d}{dx}\log\left(x^3+7x+9\right)$$

$$\Rightarrow\frac{1}{y}\frac{dy}{dx}=\frac{1}{x^2-5x+8}\cdot\frac{d}{dx}\left(x^2-5x+8\right)+\frac{1}{x^3+7x+9}\cdot\frac{d}{dx}\left(x^3+7x+9\right)$$

$$\Rightarrow\frac{dy}{dx}=y\left[\frac{1}{x^2-5x+8}\times(2x-5)+\frac{1}{x^3+7x+9}\times\left(3x^2+7\right)\right]$$

$$\Rightarrow\frac{dy}{dx}=\left(x^2-5x+8\right)\left(x^3+7x+9\right)\left[\frac{2x-5}{x^2-5x+8}+\frac{3x^2+7}{x^3+7x+9}\right]$$

$$\Rightarrow\frac{dy}{dx}=\left(x^2-5x+8\right)\left(x^3+7x+9\right)\left[\frac{(2x-5)\left(x^3+7x+9\right)+\left(3x^2+7\right)\left(x^2-5x+8\right)}{\left(x^2-5x+8\right)\left(x^3+7x+9\right)}\right]$$

$$\Rightarrow\frac{dy}{dx}=2x\left(x^3+7x+9\right)-5\left(x^3+7x+9\right)+3x^2\left(x^2-5x+8\right)+7\left(x^2-5x+8\right)$$

$$\Rightarrow\frac{dy}{dx}=\left(2x^4+14x^2+18x\right)-5x^3-35x-45+\left(3x^4-15x^3+24x^2\right)+\left(7x^2-35x+56\right)$$

$$\Rightarrow\frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$$

From the above three observations, it can be concluded that all the results of $\frac{dy}{dx}$ are same.

Question 18:

If u, v and w are functions of x, then show that

in two ways-first by repeated application of product rule, second by logarithmic differentiation.

Answer

Let $y=u.v.w=u.(v.w)$

By applying product rule, we obtain

$$\frac{dy}{dx}=\frac{du}{dx}.(v.w)+u.\frac{d}{dx}(v\cdot w)$$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}v\cdot w+u\left[\frac{dv}{dx}\cdot w+v\cdot\frac{dw}{dx}\right]\qquad\text{(Again applying product rule)}$$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}\cdot v\cdot w+u\cdot\frac{dv}{dx}\cdot w+u\cdot v\cdot\frac{dw}{dx}$$

By taking logarithm on both sides of the equation $y=u.v.w$, we obtain

$$\log y=\log u+\log v+\log w$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d}{dx}(\log u)+\frac{d}{dx}(\log v)+\frac{d}{dx}(\log w)$$

$$\Rightarrow\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}$$

$$\Rightarrow\frac{dy}{dx}=y\left(\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}\right)$$

$$\Rightarrow\frac{dy}{dx}=u.v.w.\left(\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}\right)$$

$$\therefore\frac{dy}{dx}=\frac{du}{dx}\cdot v\cdot w+u\cdot\frac{dv}{dx}\cdot w+u\cdot v\cdot\frac{dw}{dx}$$

Exercise 5.6

Question 1:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = 2at^2$, $y = at^4$

Answer

The given equations are $x = 2at^2$ and $y = at^4$

$$\text{Then, } \frac{dx}{dt} = \frac{d}{dt}\left(2at^2\right) = 2a \cdot \frac{d}{dt}\left(t^2\right) = 2a \cdot 2t = 4at$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(at^4\right) = a \cdot \frac{d}{dt}\left(t^4\right) = a \cdot 4 \cdot t^3 = 4at^3$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4at^3}{4at} = t^2$$

Question 2:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a \cos \theta$, $y = b \cos \theta$

Answer

The given equations are $x = a \cos \theta$ and $y = b \cos \theta$

Question 3:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

x = sin t, y = cos 2t

Answer

The given equations are x = sin t and y = cos 2t

$$\frac{dx}{dt}=\frac{d}{dt}(4t)=4$$

$$\frac{dy}{dt}=\frac{d}{dt}\left(\frac{4}{t}\right) \quad \left(\frac{1}{t}\right)=4\cdot\left(\frac{-1}{t^2}\right)=\frac{-4}{t^2}$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\left(\frac{-4}{t^2}\right)}{4}=\frac{-1}{t^2}$$

$$\text{Then, } \frac{dx}{dt}=\frac{d}{dt}(\sin t)=\cos t$$

$$\frac{dy}{dt}=\frac{d}{dt}(\cos 2t)=-\sin 2t\cdot\frac{d}{dt}(2t)=-2\sin 2t$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{-2\sin 2t}{\cos t}=\frac{-2\cdot 2\sin t\cos t}{\cos t}=-4\sin t$$

Question 4

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

Answer

The given equations are $x=4t$ and $y=\frac{4}{t}$

Question 5:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

Answer

The given equations are $x = \cos\theta - \cos 2\theta$ and $y = \sin\theta - \sin 2\theta$

Question 6:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

Answer

The given equations are $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$

$$\text{Then, } \frac{dx}{d\theta} = a\left[\frac{d}{d\theta}(\theta) - \frac{d}{d\theta}(\sin\theta)\right] = a(1 - \cos\theta)$$

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(1) + \frac{d}{d\theta}(\cos\theta)\right] = a\left[0 + (-\sin\theta)\right] = -a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-a\sin\theta}{a(1-\cos\theta)} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

Question 7:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$

Answer

The given equations are $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

Then, $\frac{dx}{dt}=\frac{d}{dt}\left[\frac{\sin^3 t}{\sqrt{\cos 2t}}\right]$

$$=\frac{\sqrt{\cos 2t}\cdot\frac{d}{dt}\left(\sin^3 t\right)-\sin^3 t\cdot\frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$=\frac{\sqrt{\cos 2t}\cdot 3\sin^2 t\cdot\frac{d}{dt}(\sin t)-\sin^3 t\times\frac{1}{2\sqrt{\cos 2t}}\cdot\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$=\frac{3\sqrt{\cos 2t}\cdot\sin^2 t\cos t-\frac{\sin^3 t}{2\sqrt{\cos 2t}}.(-2\sin 2t)}{\cos 2t}$$

$$=\frac{3\cos 2t\sin^2 t\cos t+\sin^3 t\sin 2t}{\cos 2t\sqrt{\cos 2t}}$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{-3\cos 2t.\cos^2 t.\sin t+\cos^3 t\sin 2t}{3\cos 2t\sin^2 t\cos t+\sin^3 t\sin 2t}$$

$$=\frac{-3\cos 2t.\cos^2 t.\sin t+\cos^3 t\left(2\sin t\cos t\right)}{3\cos 2t\sin^2 t\cos t+\sin^3 t\left(2\sin t\cos t\right)}$$

$$=\frac{\sin t\cos t\left[-3\cos 2t.\cos t+2\cos^3 t\right]}{\sin t\cos t\left[3\cos 2t\sin t+2\sin^3 t\right]}$$

$$=\frac{\left[-3\left(2\cos^2 t-1\right)\cos t+2\cos^3 t\right]}{\left[3\left(1-2\sin^2 t\right)\sin t+2\sin^3 t\right]} \qquad \begin{bmatrix}\cos 2t=\left(2\cos^2 t-1\right),\\ \cos 2t=\left(1-2\sin^2 t\right)\end{bmatrix}$$

$$=\frac{-4\cos^3 t+3\cos t}{3\sin t-4\sin^3 t}$$

Question 8:

$$=\frac{-\cos 3t}{\sin 3t} \qquad \begin{bmatrix}\cos 3t=4\cos^3 t-3\cos t,\\ \sin 3t=3\sin t-4\sin^3 t\end{bmatrix}$$

$$=-\cot 3t$$

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

Answer

The given equations are $x=a\left(\cos t+\log\tan\frac{t}{2}\right)$ and $y=a\sin t$

$$\text{Then, } \frac{dx}{dt} = a \cdot \left[\frac{d}{dt}(\cos t) + \frac{d}{dt}\left(\log \tan \frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \frac{d}{dt}\left(\tan \frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \cot \frac{t}{2} \cdot \sec^2 \frac{t}{2} \cdot \frac{d}{dt}\left(\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}} \times \frac{1}{\cos^2 \frac{t}{2}} \times \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}}\right]$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\left(\frac{-\sin^2 t + 1}{\sin t}\right)$$

$$= a \frac{\cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = a \frac{d}{dt}(\sin t) = a \cos t$$

Question 9:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

Answer

The given equations are $x = a\sec\theta$ and $y = b\tan\theta$

$$\text{Then, } \frac{dx}{d\theta} = a\cdot\frac{d}{d\theta}(\sec\theta) = a\sec\theta\tan\theta$$

$$\frac{dy}{d\theta} = b\cdot\frac{d}{d\theta}(\tan\theta) = b\sec^2\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\sec\theta\cot\theta = \frac{b\cos\theta}{a\cos\theta\sin\theta} = \frac{b}{a}\times\frac{1}{\sin\theta} = \frac{b}{a}\operatorname{cosec}\theta$$

Question 10:

If x and y are connected parametrically by the equation, without eliminating the parameter, find $\frac{dy}{dx}$.

$x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta)$

Answer

The given equations are

$$x = a(\cos\theta + \theta\sin\theta) \text{ and } y = a(\sin\theta - \theta\cos\theta)$$

$$\text{Then, } \frac{dx}{d\theta} = a\left[\frac{d}{d\theta}\cos\theta + \frac{d}{d\theta}(\theta\sin\theta)\right] = a\left[-\sin\theta + \theta\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(\theta)\right]$$

$$= a[-\sin\theta + \theta\cos\theta + \sin\theta] = a\theta\cos\theta$$

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\theta\cos\theta)\right] = a\left[\cos\theta - \left\{\theta\frac{d}{d\theta}(\cos\theta) + \cos\theta\cdot\frac{d}{d\theta}(\theta)\right\}\right]$$

$$= a[\cos\theta + \theta\sin\theta - \cos\theta]$$

$$= a\theta\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

Question 11:

If $x=\sqrt{a^{\sin^{-1}t}},\ y=\sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx}=-\frac{y}{x}$

Answer

The given equations are $x=\sqrt{a^{\sin^{-1}t}}$ and $y=\sqrt{a^{\cos^{-1}t}}$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\left(\frac{-y\log a}{2\sqrt{1-t^2}}\right)}{\left(\frac{x\log a}{2\sqrt{1-t^2}}\right)}=-\frac{y}{x}.$$

Hence, proved.

$x^2 + 3x + 2$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(3x) + \frac{d}{dx}(2) = 2x + 3 + 0 = 2x + 3$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(2x+3) = \frac{d}{dx}(2x) + \frac{d}{dx}(3) = 2 + 0 = 2$$

Exercise 5.7

Question 1:

Find the second order derivatives of the function.

Answer

Let $y = x^2 + 3x + 2$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{20}\right) = 20x^{19}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(20x^{19}\right) = 20\frac{d}{dx}\left(x^{19}\right) = 20 \cdot 19 \cdot x^{18} = 380x^{18}$$

Question 2:

Find the second order derivatives of the function.

Answer

Let $y = x\cos x$ x^{20}

Then,

Question 3:

Find the second order derivatives of the function.

Answer

Let $y = x \cdot \cos x$

Then,

$\log x$

$$\frac{dy}{dx} = \frac{d}{dx}(x \cdot \cos x) = \cos x \cdot \frac{d}{dx}(x) + x\frac{d}{dx}(\cos x) = \cos x \cdot 1 + x(-\sin x) = \cos x - x\sin x$$

$$\frac{dy}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x} \quad \sin x] = \frac{d}{dx}(\cos x) - \frac{d}{dx}(x\sin x)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2} \quad x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\sin x)\Big]$$

Question 4:

$$= -\sin x - (\sin x + x\cos x)$$

Find the second order derivatives of the function.

$$= -(x\cos x + 2\sin x)$$

Answer

Let $x^3 \log x$ $\log x$

Then,

Question 5:

Find the second order derivatives of the function. $x^3 \log x$

Answer

Let $y = x^3 \log x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^3 \log x\right] = \log x \cdot \frac{d}{dx}\left(x^3\right) + x^3 \cdot \frac{d}{dx}(\log x)$$

$$= \log x \cdot 3x^2 + x^3 \cdot \frac{1}{x} = \log x \cdot 3x^2 + x^2$$

$$= x^2\left(1 + 3\log x\right)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[x^2\left(1+3\log x\right)\right]$$

$$= \left(1+3\log x\right)\cdot\frac{d}{dx}\left(x^2\right) + x^2\frac{d}{dx}\left(1+3\log x\right)$$

$$= \left(1+3\log x\right)\cdot 2x + x^2 \cdot \frac{3}{x}$$

$$= 2x + 6x\log x + 3x$$

$$= 5x + 6x\log x$$

$$= x\left(5 + 6\log x\right)$$

Question 6:

Find the second order derivatives of the function. $e^x \sin 5x$

Answer

Let $y = e^x \sin 5x$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x \sin 5x\right) = \sin 5x \cdot \frac{d}{dx}\left(e^x\right) + e^x \frac{d}{dx}(\sin 5x)$$

$$= \sin 5x \cdot e^x + e^x \cdot \cos 5x \cdot \frac{d}{dx}(5x) = e^x \sin 5x + e^x \cos 5x \cdot 5$$

$$= e^x\left(\sin 5x + 5\cos 5x\right)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[e^x\left(\sin 5x + 5\cos 5x\right)\right]$$

$$= \left(\sin 5x + 5\cos 5x\right)\cdot\frac{d}{dx}\left(e^x\right) + e^x \cdot \frac{d}{dx}\left(\sin 5x + 5\cos 5x\right)$$

$$= \left(\sin 5x + 5\cos 5x\right)e^x + e^x\left[\cos 5x \cdot \frac{d}{dx}(5x) + 5(-\sin 5x)\cdot\frac{d}{dx}(5x)\right]$$

$$= e^x\left(\sin 5x + 5\cos 5x\right) + e^x\left(5\cos 5x - 25\sin 5x\right)$$

$$= e^x\left(10\cos 5x - 24\sin 5x\right) = 2e^x\left(5\cos 5x - 12\sin 5x\right)$$

Then,

Question 7:

Find the second order derivatives of the function.

$e^{6x}\cos 3x$

Answer

Let $y = e^{6x}\cos 3x$

Then,

Question 8:

Find the second order derivatives of the function.

Answer

Let $y = \tan^{-1} x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin(\log x)\right] = \cos(\log x)\cdot\frac{d}{dx}(\log x) = \frac{\cos(\log x)}{x}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right]$$

$$+x^2)$$

$$= \frac{x\cdot\frac{d}{dx}\left[\cos(\log x)\right] - \cos(\log x)\cdot\frac{d}{dx}(x)}{x^2}$$

Question 9:

Find the second order derivatives of the function.

$$= \frac{x\left[-\sin(\log x)\cdot\frac{d}{dx}(\log x)\right] - \cos(\log x).1}{x^2}$$

$$\log(\log x)$$

Answer $= \frac{-x\sin(\log x).\frac{1}{x} - \cos(\log x)}{x^2}$

Let $y = \log(\log x)$ $\frac{\left[\ldots g x) + \cos(\log x)\right]}{x^2}$

Then,

Question 10:

Find the second order derivatives of the function.

Answer

Let $y = \sin(\log x)$

Then,

Question 11:

If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$

Answer

It is given that, $y = 5\cos x - 3\sin x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(5\cos x) - \frac{d}{dx}(3\sin x) = 5\frac{d}{dx}(\cos x) - 3\frac{d}{dx}(\sin x)$$

$$= 5(-\sin x) - 3\cos x = -(5\sin x + 3\cos x)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left[-(5\sin x + 3\cos x)\right]$$

$$= -\left[5\cdot\frac{d}{dx}(\sin x) + 3\cdot\frac{d}{dx}(\cos x)\right]$$

$$= -\left[5\cos x + 3(-\sin x)\right]$$

$$= -\left[5\cos x - 3\sin x\right]$$

$$= -y$$

$$\therefore \frac{d^2y}{dx^2} + y = 0$$

Hence, proved.

Question 12:

If $y=\cos^{-1}x$, find $\frac{d^2y}{dx^2}$ in terms of y alone.

Answer

It is given that, $y=\cos^{-1}x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\cos^{-1}x\right)=\frac{-1}{\sqrt{1-x^2}}=-\left(1-x^2\right)^{\frac{-1}{2}}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[-\left(1-x^2\right)^{\frac{-1}{2}}\right]$$

$$=-\left(-\frac{1}{2}\right)\cdot\left(1-x^2\right)^{\frac{-3}{2}}\cdot\frac{d}{dx}\left(1-x^2\right)$$

$$=\frac{1}{2\sqrt{\left(1-x^2\right)^3}}\times(-2x)$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{-x}{\sqrt{\left(1-x^2\right)^3}}\qquad\text{...(i)}$$

$y=\cos^{-1}x\Rightarrow x=\cos y$

Putting $x=\cos y$ in equation (i), we obtain

$$\frac{d^2y}{dx^2}=\frac{-\cos y}{\sqrt{\left(1-\cos^2 y\right)^3}}$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{-\cos y}{\sqrt{\left(\sin^2 y\right)^3}}$$

$$=\frac{-\cos y}{\sin^3 y}$$

$$=\frac{-\cos y}{\sin y}\times\frac{1}{\sin^2 y}$$

$$\Rightarrow\frac{d^2y}{dx^2}=-\cot y\cdot\operatorname{cosec}^2 y$$

Question 13:

If $y=3\cos(\log x)+4\sin(\log x)$, show that $x^2y_2+xy_1+y=0$

Answer

It is given that, $y=3\cos(\log x)+4\sin(\log x)$

Then,

$$y_1 = 3 \cdot \frac{d}{dx}\left[\cos(\log x)\right] + 4 \cdot \frac{d}{dx}\left[\sin(\log x)\right]$$

$$= 3 \cdot \left[-\sin(\log x) \cdot \frac{d}{dx}(\log x)\right] + 4 \cdot \left[\cos(\log x) \cdot \frac{d}{dx}(\log x)\right]$$

$$\therefore y_1 = \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x} = \frac{4\cos(\log x) - 3\sin(\log x)}{x}$$

$$\therefore y_2 = \frac{d}{dx}\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right)$$

$$= \frac{x\{4\cos(\log x) - 3\sin(\log x)\}' - \{4\cos(\log x) - 3\sin(\log x)\}(x)'}{x^2}$$

$$= \frac{x\left[4\{\cos(\log x)\}' - 3\{\sin(\log x)\}'\right] - \{4\cos(\log x) - 3\sin(\log x)\}.1}{x^2}$$

$$= \frac{x\left[-4\sin(\log x).(\log x)' - 3\cos(\log x).(\log x)'\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{x\left[-4\sin(\log x).\frac{1}{x} - 3\cos(\log x).\frac{1}{x}\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-4\sin(\log x) - 3\cos(\log x) - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-\sin(\log x) - 7\cos(\log x)}{x^2}$$

$$\therefore x^2 y_2 + x y_1 + y$$

$$= x^2\left(\frac{-\sin(\log x) - 7\cos(\log x)}{x^2}\right) + x\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right) + 3\cos(\log x) + 4\sin(\log x)$$

$$= -\sin(\log x) - 7\cos(\log x) + 4\cos(\log x) - 3\sin(\log x) + 3\cos(\log x) + 4\sin(\log x)$$

$$= 0$$

Hence, proved.

Question 14:

If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

Answer

It is given that, $y = Ae^{mx} + Be^{nx}$

Then,

$$\frac{dy}{dx} = A\cdot\frac{d}{dx}\left(e^{mx}\right) + B\cdot\frac{d}{dx}\left(e^{nx}\right) = A\cdot e^{mx}\cdot\frac{d}{dx}(mx) + B\cdot e^{nx}\cdot\frac{d}{dx}(nx) = Ame^{mx} + Bne^{nx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(Ame^{mx} + Bne^{nx}\right) = Am\cdot\frac{d}{dx}\left(e^{mx}\right) + Bn\cdot\frac{d}{dx}\left(e^{nx}\right)$$

$$= Am\cdot e^{mx}\cdot\frac{d}{dx}(mx) + Bn\cdot e^{nx}\cdot\frac{d}{dx}(nx) = Am^2e^{mx} + Bn^2e^{nx}$$

$$\therefore \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$$

$$= Am^2e^{mx} + Bn^2e^{nx} - (m+n)\cdot\left(Ame^{mx} + Bne^{nx}\right) + mn\left(Ae^{mx} + Be^{nx}\right)$$

$$= Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Bmne^{nx} - Amne^{mx} - Bn^2e^{nx} + Amne^{mx} + Bmne^{nx}$$

$$= 0$$

Hence, proved.

Question 15:

If $y = 500e^{7x} + 600e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$

Answer

It is given that, $y = 500e^{7x} + 600e^{-7x}$

Then,

$$\frac{dy}{dx} = 500.\frac{d}{dx}\left(e^{7x}\right) + 600.\frac{d}{dx}\left(e^{-7x}\right)$$
$$= 500 \cdot e^{7x} \cdot \frac{d}{dx}(7x) + 600 \cdot e^{-7x} \cdot \frac{d}{dx}(-7x)$$
$$= 3500e^{7x} - 4200e^{-7x}$$
$$\therefore \frac{d^2y}{dx^2} = 3500 \cdot \frac{d}{dx}\left(e^{7x}\right) - 4200.\frac{d}{dx}\left(e^{-7x}\right)$$
$$= 3500 \cdot e^{7x}.\frac{d}{dx}(7x) - 4200 \cdot e^{-7x} \cdot \frac{d}{dx}(-7x)$$
$$= 7 \times 3500 \cdot e^{7x} + 7 \times 4200 \cdot e^{-7x}$$
$$= 49 \times 500e^{7x} + 49 \times 600e^{-7x}$$
$$= 49\left(500e^{7x} + 600e^{-7x}\right)$$
$$= 49y$$

Hence, proved.

Question 16:

If $e^y(x+1) = 1$, show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

Answer

The given relationship is $e^y(x+1) = 1$

Taking logarithm on both the sides, we obtain

Differentiating this relationship with respect to x, we obtain

$$\frac{dy}{dx}=(x+1)\frac{d}{dx}\left(\frac{1}{x+1}\right)=(x+1).\frac{-1}{(x+1)^2}=\frac{-1}{x+1}$$

$$\therefore \frac{d^2y}{dx^2}=-\frac{d}{dx}\left(\frac{1}{x+1}\right)=-\left(\frac{-1}{(x+1)^2}\right)=\frac{1}{(x+1)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{-1}{x+1}\right)^2$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$$

Hence, proved.

Question 17:

If $y=\left(\tan^{-1}x\right)^2$, show that $\left(x^2+1\right)^2 y_2+2x\left(x^2+1\right)y_1=2$

Answer

The given relationship is $y=\left(\tan^{-1}x\right)^2$

Then,

Hence, proved.

Exercise 5.8

Question 1:

Verify Rolle's Theorem for the function $f(x)=x^2+2x-8$, $x\in[-4,2]$

Answer

$$f(-4)=(-4)^2+2\times(-4)-8=16-8-8=0$$
$$f(2)=(2)^2+2\times2-8=4+4-8=0$$

The given function, $f(x)=x^2+2x-8$, being a polynomial function, is continuous in [−4, 2] and is differentiable in (−4, 2).

∴ f (−4) = f (2) = 0

⇒ The value of f (x) at −4 and 2 coincides.

Rolle's Theorem states that there is a point c ∈ (−4, 2) such that $f'(c)=0$

Hence, Rolle's Theorem is verified for the given function.

Question 2:

Examine if Rolle's Theorem is applicable to any of the following functions. Can you say some thing about the converse of Rolle's Theorem from these examples?

(i) $f(x)=[x]$ for $x\in[5, 9]$

(ii) $f(x)=[x]$ for $x\in[-2, 2]$

(iii) $f(x)=x^2-1$ for $x\in[1, 2]$

Answer

By Rolle's Theorem, for a function $f:[a, b]\to \mathbf{R}$, if

(a) f is continuous on [a, b]

(b) f is differentiable on (a, b)

(c) f (a) = f (b)

then, there exists some c ∈ (a, b) such that $f'(c)=0$

Therefore, Rolle's Theorem is not applicable to those functions that do not satisfy any of the three conditions of the hypothesis.

(i) $f(x)=[x]$ for $x\in[5, 9]$

It is evident that the given function f (x) is not continuous at every integral point.

In particular, f(x) is not continuous at x = 5 and x = 9

⇒ f (x) is not continuous in [5, 9].

The differentiability of f in (5, 9) is checked as follows.

Let n be an integer such that n ∈ (5, 9).

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-n}{h}=\lim_{h\to 0^+}0=0$$

Since the left and right hand limits of f at x = n are not equal, f is not differentiable at x = n

∴f is not differentiable in (5, 9).

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x)=[x]$ for $x\in[5,\ 9]$.

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

It is evident that the given function f (x) is not continuous at every integral point.
In particular, f(x) is not continuous at x = −2 and x = 2

⇒ f (x) is not continuous in [−2, 2].

The differentiability of f in (−2, 2) is checked as follows.

Let n be an integer such that n ∈ (−2, 2).

The left hand limit of f at $x = n$ is,

$$\lim_{h\to0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to0^-}\frac{n-1-n}{h}=\lim_{h\to0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to0^+}\frac{n-n}{h}=\lim_{h\to0^+}0=0$$

Since the left and right hand limits of f at x = n are not equal, f is not differentiable at x = n

∴f is not differentiable in (−2, 2).

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x)=[x]$ for $x\in[-2,\ 2]$.

(iii) $f(x)=x^2-1$ for $x\in[1,\ 2]$

It is evident that f, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

∴f (1) ≠ f (2)

It is observed that f does not satisfy a condition of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x)=x^2-1$ for $x\in[1,\ 2]$.

Question 3:

If $f:[-5,5]\to\mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5)\neq f(5)$.

Answer

It is given that $f:[-5,5]\to \mathbf{R}$ is a differentiable function.

Since every differentiable function is a continuous function, we obtain

(a) f is continuous on [−5, 5].

(b) f is differentiable on (−5, 5).

Therefore, by the Mean Value Theorem, there exists c ∈ (−5, 5) such that

It is also given that $f'(x)$ does not vanish anywhe

Hence, proved.

Question 4:

Verify Mean Value Theorem, if $f(x)=x^2-4x-3$ in the interval $[a,\ b]$, where $a=1$ and $b=4$.

Answer

The given function is $f(x)=x^2-4x-3$

f, being a polynomial function, is continuous in [1, 4] and is differentiable in (1, 4) whose derivative is 2x − 4.

Mean Value Theorem states that there is a point c ∈ (1, 4) such that $f'(c)=1$

Hence, Mean Value Theorem is verified for the given function.

Question 5:

Verify Mean Value Theorem, if $f(x)=x^3-5x^2-3x$ in the interval [a, b], where a = 1 and b = 3. Find all $c\in(1,3)$ for which $f'(c)=0$

Answer

The given function f is $f(x)=x^3-5x^2-3x$

f, being a polynomial function, is continuous in [1, 3] and is differentiable in (1, 3) whose derivative is $3x^2 - 10x - 3$.

Mean Value Theorem states that there exist a point c ∈ (1, 3) such that $f'(c)=-10$

$$
\begin{aligned}
&f'(c)=-10\\
&\Rightarrow 3c^2-10c-3=10\\
&\Rightarrow 3c^2-10c+7=0\\
&\Rightarrow 3c^2-3c-7c+7=0\\
&\Rightarrow 3c(c-1)-7(c-1)=0\\
&\Rightarrow (c-1)(3c-7)=0\\
&\Rightarrow c=1,\ \frac{7}{3},\text{ where } c=\frac{7}{3}\in(1,\ 3)
\end{aligned}
$$

Hence, Mean Value Theorem is verified for the given function and $c=\frac{7}{3}\in(1,\ 3)$ is the only point for which $f'(c)=0$

Question 6:

Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

Answer

Mean Value Theorem states that for a function $f:[a,\ b]\to\mathbf{R}$, if

(a) f is continuous on [a, b]

(b) f is differentiable on (a, b)

then, there exists some c ∈ (a, b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$

Therefore, Mean Value Theorem is not applicable to those functions that do not satisfy any of the two conditions of the hypothesis.

(i)

It is evident that the given function f (x) is not continuous at every integral point.

In particular, f(x) is not continuous at x = 5 and x = 9

$f(x)=[x]$ for $x\in[5,\ 9]$

⇒ f (x) is not continuous in [5, 9].

The differentiability of f in (5, 9) is checked as follows.

Let n be an integer such that n ∈ (5, 9).

Since the left and right hand limits of f at x = n are not equal, f is not differentiable at x = n

∴f is not differentiable in (5, 9).

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x\in[5,\ 9]$.

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

It is evident that the given function f (x) is not continuous at every integral point.

In particular, f(x) is not continuous at x = −2 and x = 2

⇒ f (x) is not continuous in [−2, 2].

The differentiability of f in (−2, 2) is checked as follows.

Let n be an integer such that n ∈ (−2, 2).

$f(1)=1^2-1=0,\ f(2)=2^2-1=3$

$$\therefore \frac{f(b)-f(a)}{b-a}=\frac{f(2)-f(1)}{2-1}=\frac{3-0}{1}=3$$

Since the left and right hand limits of f at x = n are not equal, f is not differentiable at x = n

∴f is not differentiable in (−2, 2).

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x\in[-2,\ 2]$.

(iii) $f(x)=x^2-1$ for $x\in[1,\ 2]$

It is evident that f, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

It is observed that f satisfies all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is applicable for $f(x)=x^2-1$ for $x\in[1,\ 2]$.

It can be proved as follows.

$$f'(x) = 2x$$
$$\therefore f'(c) = 3$$
$$\Rightarrow 2c = 3$$
$$\Rightarrow c = \frac{3}{2} = 1.5, \text{ where } 1.5 \in [1, 2]$$

$\left(3x^2-9x+5\right)^9$

Question 1:

$\text{Let } y=\left(3x^2-9x+5\right)^9$

Answer

Using chain rule, we obtain

Question 2:

Answer

Question 3:

Answer

Let $y=(5x)^{3\cos 2x}$

Taking logarithm on both the sides, we obtain

$\log y = 3\cos 2x \log 5x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = 3\left[\log 5x \cdot \frac{d}{dx}(\cos 2x) + \cos 2x \cdot \frac{d}{dx}(\log 5x)\right]$$

$$\Rightarrow \frac{dy}{dx} = 3y\left[\log 5x(-\sin 2x)\cdot\frac{d}{dx}(2x) + \cos 2x \cdot \frac{1}{5x}\cdot\frac{d}{dx}(5x)\right]$$

$$\Rightarrow \frac{dy}{dx} = 3y\left[-2\sin 2x \log 5x + \frac{\cos 2x}{x}\right]$$

$$\Rightarrow \frac{dy}{dx} = 3y\left[\frac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$$

$$\therefore \frac{dy}{dx} = (5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$$

Question 4:

$\sin^{-1}\left(x\sqrt{x}\right),\ 0 \le x \le 1$

Answer

Let $y = \sin^{-1}\left(x\sqrt{x}\right)$

Using chain rule, we obtain

$$\frac{dy}{dx}=\frac{d}{dx}\sin^{-1}\left(x\sqrt{x}\right)$$

$$=\frac{1}{\sqrt{1-\left(x\sqrt{x}\right)^{2}}}\times\frac{d}{dx}\left(x\sqrt{x}\right)$$

$$=\frac{1}{\sqrt{1-x^{3}}}\cdot\frac{d}{dx}\left(x^{\frac{3}{2}}\right)$$

$$=\frac{1}{\sqrt{1-x^{3}}}\times\frac{3}{2}\cdot x^{\frac{1}{2}}$$

$$=\frac{3\sqrt{x}}{2\sqrt{1-x^{3}}}$$

$$=\frac{3}{2}\sqrt{\frac{x}{1-x^{3}}}$$

Question 5:

Answer

Let $y=\dfrac{\cos^{-1}\dfrac{x}{2}}{\sqrt{2x+7}}$

By quotient rule, we obtain

$$\frac{dy}{dx}=\frac{\sqrt{2x+7}\,\dfrac{d}{dx}\left(\cos^{-1}\dfrac{x}{2}\right)-\left(\cos^{-1}\dfrac{x}{2}\right)\dfrac{d}{dx}\left(\sqrt{2x+7}\right)}{\left(\sqrt{2x+7}\right)^2}$$

$$=\frac{\sqrt{2x+7}\left[\dfrac{-1}{\sqrt{1-\left(\dfrac{x}{2}\right)^2}}\cdot\dfrac{d}{dx}\left(\dfrac{x}{2}\right)\right]-\left(\cos^{-1}\dfrac{x}{2}\right)\dfrac{1}{2\sqrt{2x+7}}\cdot\dfrac{d}{dx}(2x+7)}{2x+7}$$

$$=\frac{\sqrt{2x+7}\,\dfrac{-1}{\sqrt{4-x^2}}-\left(\cos^{-1}\dfrac{x}{2}\right)\dfrac{2}{2\sqrt{2x+7}}}{2x+7}$$

$$=\frac{-\sqrt{2x+7}}{\sqrt{4-x^2}\times(2x+7)}-\frac{\cos^{-1}\dfrac{x}{2}}{\left(\sqrt{2x+7}\right)(2x+7)}$$

Question 6:

$$=-\left[\frac{1}{\sqrt{4-x^2}\sqrt{2x+7}}+\frac{\cos^{-1}\dfrac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$$

Answer

$$\text{Let } y=\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right] \qquad \ldots(1)$$

$$\text{Then, } \frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$$

$$=\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)}$$

$$=\frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1-\sin x)(1+\sin x)}}{(1+\sin x)-(1-\sin x)}$$

$$=\frac{2+2\sqrt{1-\sin^2 x}}{2\sin x}$$

$$=\frac{1+\cos x}{\sin x}$$

$$=\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\cot\frac{x}{2}$$

Therefore, equation (1) becomes

$$y=\cot^{-1}\left(\cot\frac{x}{2}\right)$$

$$\Rightarrow y=\frac{x}{2}$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\frac{d}{dx}(x)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}$$

Question 7:

$$(\log x)^{\log x},\ x>1$$

Answer

$$\text{Let } y=(\log x)^{\log x}$$

Taking logarithm on both the sides, we obtain

$$\log y = \log x \cdot \log(\log x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[\log x \cdot \log(\log x)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\log x).\frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\log(\log x).\frac{1}{x} + \log x \cdot \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x}\log(\log x) + \frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\log x}\left[\frac{1}{x} + \frac{\log(\log x)}{x}\right]$$

Differentiating both sides with respect to x, we obtQuestion 8:

$\cos(a\cos x + b\sin x)$, for some constant a and b.

Answer

By using chain rule, we obtain

Question 9:

Answer

Let $y=(\sin x-\cos x)^{(\sin x-\cos x)}$

Taking logarithm on both the sides, we obtain

$$\log y=\log\left[(\sin x-\cos x)^{(\sin x-\cos x)}\right]$$
$$\Rightarrow \log y=(\sin x-\cos x)\cdot\log(\sin x-\cos x)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\left[(\sin x-\cos x)\log(\sin x-\cos x)\right]$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\log(\sin x-\cos x)\cdot\frac{d}{dx}(\sin x-\cos x)+(\sin x-\cos x)\cdot\frac{d}{dx}\log(\sin x-\cos x)$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\log(\sin x-\cos x)\cdot(\cos x+\sin x)+(\sin x-\cos x)\cdot\frac{1}{(\sin x-\cos x)}\cdot\frac{d}{dx}(\sin x-\cos x)$$
$$\Rightarrow \frac{dy}{dx}=(\sin x-\cos x)^{(\sin x-\cos x)}\left[(\cos x+\sin x)\cdot\log(\sin x-\cos x)+(\cos x+\sin x)\right]$$
$$\therefore \frac{dy}{dx}=(\sin x-\cos x)^{(\sin x-\cos x)}(\cos x+\sin x)\left[1+\log(\sin x-\cos x)\right]$$

Differentiating both sides with respect to x, we obtain

Question 10:

$x^x+x^a+a^x+a^a$, for some fixed $a>0$ and $x>0$

Answer

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\log x\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{du}{dx}=u\left[\log x\cdot 1+x\cdot\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=x^x\left[\log x+1\right]=x^x\left(1+\log x\right) \qquad ...(2)$$

$$v=x^a$$

$$\therefore\frac{dv}{dx}=\frac{d}{dx}\left(x^a\right)$$

$$\Rightarrow\frac{dv}{dx}=ax^{a-1} \qquad ...(3)$$

$$w=a^x$$

$$\Rightarrow\log w=\log a^x$$

$$\Rightarrow\log w=x\log a$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{w}\cdot\frac{dw}{dx}=\log a\cdot\frac{d}{dx}(x)$$

$$\Rightarrow\frac{dw}{dx}=w\log a$$

$$\Rightarrow\frac{dw}{dx}=a^x\log a \qquad ...(4)$$

$s = a^a$

Since a is constant, a^a is also a constant.

$$\therefore\frac{ds}{dx}=0 \qquad ...(5)$$

From (1), (2), (3), (4), and (5), we obtain

Question 11:

$x^{x^2-3}+(x-3)^{x^2}$, for $x>3$

Answer

Let $y=x^{x^2-3}+(x-3)^{x^2}$

Also, let $u=x^{x^2-3}$ and $v=(x-3)^{x^2}$

$\therefore y=u+v$

Differentiating both sides with respect to x, we obtain

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad ...(1)$$

$u=x^{x^2-3}$

$\therefore \log u=\log\left(x^{x^2-3}\right)$

$\log u=\left(x^2-3\right)\log x$

Differentiating with respect to x, we obtain

$$\frac{1}{u}\cdot\frac{du}{dx}=\log x\cdot\frac{d}{dx}\left(x^2-3\right)+\left(x^2-3\right)\cdot\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx}=\log x\cdot 2x+\left(x^2-3\right)\cdot\frac{1}{x}$$

$$\Rightarrow \frac{du}{dx}=x^{x^2-3}\cdot\left[\frac{x^2-3}{x}+2x\log x\right]$$

Also,

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\log(x-3)\cdot\frac{d}{dx}(x^2)+x^2\cdot\frac{d}{dx}[\log(x-3)]$$

$$\Rightarrow\frac{1}{v}\frac{dv}{dx}=\log(x-3)\cdot 2x+x^2\cdot\frac{1}{x-3}\cdot\frac{d}{dx}(x-3)$$

$$\Rightarrow\frac{dv}{dx}=v\left[2x\log(x-3)+\frac{x^2}{x-3}\cdot 1\right]$$

$$\Rightarrow\frac{dv}{dx}=(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$$

Substituting the expressions of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in equation (1), we obtain

$$\frac{dy}{dx}=x^{x^2-3}\left[\frac{x^2-3}{x}+2x\log x\right]+(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$$

Question 12:

Find $\frac{dy}{dx}$, if $y=12(1-\cos t), x=10(t-\sin t), -\frac{\pi}{2}<t<\frac{\pi}{2}$

Answer

Question 13:

Find $\frac{dy}{dx}$, if $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2},\ -1\le x\le 1$

Answer

It is given that, $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left[\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(\sin^{-1}x\right)+\frac{d}{dx}\left(\sin^{-1}\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\cdot\frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{x}\cdot\frac{1}{2\sqrt{1-x^2}}\cdot\frac{d}{dx}\left(1-x^2\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{2x\sqrt{1-x^2}}(-2x)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx}=0$$

Question 14:

If , for, −1 < x <1, prove that

Answer

It is given that,

$$x\sqrt{1+y}+y\sqrt{1+x}=0$$

$$\frac{dy}{dx}=-\frac{1}{(1+x)^2}$$

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$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring both sides, we obtain

$x^2(1+y) = y^2(1+x)$

$\Rightarrow x^2 + x^2y = y^2 + xy^2$

$\Rightarrow x^2 - y^2 = xy^2 - x^2y$

$\Rightarrow x^2 - y^2 = xy(y-x)$

Differentiating both sides with respect to x, we obtain

$\Rightarrow (x+y)(x-y) = xy(y-x)$

$\therefore x + y = -xy$

$\Rightarrow (1+x)y = -x$

$$\Rightarrow y = \frac{-x}{(1+x)}$$

Hence, proved.

Question 15:

If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, prove that

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$ is a constant independent of a and b

Answer

It is given that, $(x-a)^2 + (y-b)^2 = c^2$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[(x-a)^2\right]+\frac{d}{dx}\left[(y-b)^2\right]=\frac{d}{dx}\left(c^2\right)$$

$$\Rightarrow 2(x-a)\cdot\frac{d}{dx}(x-a)+2(y-b)\cdot\frac{d}{dx}(y-b)=0$$

$$\Rightarrow 2(x-a)\cdot 1+2(y-b)\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(x-a)}{y-b} \qquad \ldots(1)$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{-(x-a)}{y-b}\right]$$

$$= -\left[\frac{(y-b)\cdot\frac{d}{dx}(x-a)-(x-a)\cdot\frac{d}{dx}(y-b)}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b)-(x-a)\cdot\frac{dy}{dx}}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b)-(x-a)\cdot\left\{\frac{-(x-a)}{y-b}\right\}}{(y-b)^2}\right] \qquad \text{[Using (1)]}$$

$$= -\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]$$

$$\therefore \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]} = \frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]}$$

$$= \frac{\left[\frac{c^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}} = \frac{\frac{c^3}{(y-b)^3}}{-\frac{c^2}{(y-b)^3}}$$

$= -c$, which is constant and is independent of a and b

Hence, proved.

Question 16:

If $\cos y = x\cos(a+y)$, with $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$

Answer

It is given that, $\cos y = x\cos(a+y)$

$$\therefore \frac{d}{dx}[\cos y] = \frac{d}{dx}[x\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y)\cdot\frac{d}{dx}(x) + x\cdot\frac{d}{dx}[\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y) + x\cdot[-\sin(a+y)]\frac{dy}{dx}$$

$$\Rightarrow [x\sin(a+y) - \sin y]\frac{dy}{dx} = \cos(a+y) \qquad ...(1)$$

Then, equation (1) reduces to

Since $\cos y = x\cos(a+y)$, $x = \dfrac{\cos y}{\cos(a+y)}$

$$\left[\frac{\cos y}{\cos(a+y)}\cdot\sin(a+y) - \sin y\right]\frac{dy}{dx} = \cos(a+y)$$

$$\Rightarrow [\cos y\cdot\sin(a+y) - \sin y\cdot\cos(a+y)]\cdot\frac{dy}{dx} = \cos^2(a+y)$$

$$\Rightarrow \sin(a+y-y)\frac{dy}{dx} = \cos^2(a+b)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+b)}{\sin a}$$

Hence, proved.

Question 17:

If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\dfrac{d^2y}{dx^2}$

Answer

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{at\sin t}{at\cos t} = \tan t$$

Then, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan t) = \sec^2 t \cdot \frac{dt}{dx}$

$$= \sec^2 t \cdot \frac{1}{at\cos t} \qquad \left[\frac{dx}{dt} = at\cos t \Rightarrow \frac{dt}{dx} = \frac{1}{at\cos t}\right]$$

$$= \frac{\sec^3 t}{at}, 0 < t < \frac{\pi}{2}$$

It is given that, $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$

$$\therefore \frac{dx}{dt} = a \cdot \frac{d}{dt}(\cos t + t\sin t)$$

$$= a\left[-\sin t + \sin t \cdot \frac{d}{dx}(t) + t \cdot \frac{d}{dt}(\sin t)\right]$$

$$= a[-\sin t + \sin t + t\cos t] = at\cos t$$

$$\frac{dy}{dt} = a \cdot \frac{d}{dt}(\sin t - t\cos t)$$

$$= a\left[\cos t - \left\{\cos t \cdot \frac{d}{dt}(t) + t \cdot \frac{d}{dt}(\cos t)\right\}\right]$$

$$= a[\cos t - \{\cos t - t\sin t\}] = at\sin t$$

Question 18:

If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x, and find it.

Answer

It is known that, $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Therefore, when x ≥ 0, $f(x) = |x|^3 = x^3$

In this case, $f'(x) = 3x^2$ and hence, $f''(x) = 6x$

When x < 0, $f(x) = |x|^3 = (-x)^3 = -x^3$

$$f'(x) = -3x^2$$

In this case,
and hence,

$$f''(x) = -6x$$

Thus, for $f(x)=|x|^3$, $f''(x)$ exists for all real x and is given by,

$$f''(x)=\begin{cases}6x, & \text{if } x\geq 0\\ -6x, & \text{if } x<0\end{cases}$$

Question 19:

Using mathematical induction prove that $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$ for all positive integers n.

Answer

For n = 1,

To prove: $\mathrm{P}(n):\frac{d}{dx}\left(x^n\right)=nx^{n-1}$ for all positive integers n

∴P(n) is true for n = 1

Let P(k) is true for some positive integer k.

That is, $\mathrm{P}(k):\frac{d}{dx}\left(x^k\right)=kx^{k-1}$

It has to be proved that P(k + 1) is also true.

Thus, P(k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, the statement P(n) is true for every positive integer n.

Hence, proved.

Question 20:

Using the fact that sin (A + B) = sin A cos B + cos A sin B and the differentiation, obtain the sum formula for cosines.

Answer

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

Differentiating both sides with respect to x, we obtain

Question 22:

If $y=\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, prove that $\frac{dy}{dx}=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Answer

$$y=\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

$$\Rightarrow y=(mc-nb)f(x)-(lc-na)g(x)+(lb-ma)h(x)$$

$$\text{Then, } \frac{dy}{dx}=\frac{d}{dx}\left[(mc-nb)f(x)\right]-\frac{d}{dx}\left[(lc-na)g(x)\right]+\frac{d}{dx}\left[(lb-ma)h(x)\right]$$

$$=(mc-nb)f'(x)-(lc-na)g'(x)+(lb-ma)h'(x)$$

$$=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Thus, $\dfrac{dy}{dx}=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Question 23:

If $y=e^{a\cos^{-1}x}, -1\le x\le 1$, show that $\left(1-x^2\right)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}-a^2y=0$

Answer

It is given that, $y=e^{a\cos^{-1}x}$

Taking logarithm on both the sides, we obtain

$\log y = a\cos^{-1} x \log e$

$\log y = a\cos^{-1} x$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = a \times \frac{-1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}}$$

By squaring both the sides, we obtain

$$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2}$$

$$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

Again differentiating both sides with respect to x, we obtain

$$\left(\frac{dy}{dx}\right)^2 \frac{d}{dx}(1-x^2) + (1-x^2)\times\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right] = a^2\frac{d}{dx}(y^2)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2(-2x) + (1-x^2)\times 2\frac{dy}{dx}.\frac{d^2y}{dx^2} = a^2.2y.\frac{dy}{dx}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2(-2x) + (1-x^2)\times 2\frac{dy}{dx}.\frac{d^2y}{dx^2} = a^2.2y.\frac{dy}{dx}$$

$$\Rightarrow -x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = a^2.y \qquad \left[\frac{dy}{dx} \neq 0\right]$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$

Hence, proved.